

Age-Based Preferences: Incorporating Compatible Pairs into Paired Kidney Exchange

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Abstract

The participation of compatible donor-patient pairs in Paired Kidney Exchange programs can dramatically increase the number of transplants performed. We analyze (matching) rules that provide incentives to patients with compatible donors to enroll in such programs by offering them the chance of receiving an organ with higher expected graft survival. We consider a restricted domain of patients' preferences such that each patient prefers a kidney from a compatible younger (healthier) donor to an organ from an older donor. In this setting, we identify non manipulable and efficient rules and show that they retain these properties when patients have multiple donors.

JEL: C78; D02; D78; I10.

Keywords: Kidney exchange; Age-based preferences, Priority rules, Strategy-proofness.

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1 Introduction

The literature on the allocation of indivisible objects in economies without money has found a remarkable application in the design of Paired Kidney Exchange (PKE) programs. These programs try to overcome any incompatibility (of blood or tissue types) of living donor-patient pairs by arranging swaps of donors among several pairs (Delmonico, 2004; Delmonico et al., 2004; Roth et al., 2004). Apart from the problem of finding mutually compatible donor-patient pairs, the design of PKE programs needs to address many constraints that are absent in the standard allocation problems. PKE programs usually involve the cooperation and coordination of several transplantation units at different medical centers. Logistic constraints (the use of different operations rooms, surgical teams, and the timing of the procedures) make exchanges involving too many donor patient-pairs unfeasible.¹ Real-life PKE programs have generally focused on maximizing the number of simultaneous compatible organ exchanges between two donor-patient pairs, although swaps involving more than two pairs are also carried out. To deal with situations where a donor-patient pair is involved in more than one viable exchange, PKE programs usually rely on giving priority (sequentially) to certain patients in much the same way as in the allocation of kidneys obtained from cadaveric donors.²

In spite of all their intrinsic difficulties, PKE programs are a remarkable example of how efficiency enhancement exchanges can be generated without any need for monetary compensations. It is commonly believed, however, that PKE programs could attain even better outcomes if more donor-patient pairs take part. The more donor-patient pairs there are in the pool, the higher is the likelihood of compatible exchanges being identified

¹There is growing interest in the creation of non-simultaneous, extended altruistic-donor chains that try avoid such limitations. A donor chain starts with an altruistic donor willing to donate to anyone needing a kidney transplant without having a related recipient (Ausubel and Morrill, 2010; Rees et al., 2009; Ünver, 2010).

²This is the case of the New England Paired Kidney Exchange NEPKKE (Roth et al., 2005*a,b*). Other centralized kidney exchange programs implemented in various countries, e. g. Korea, the Netherlands, the United Kingdom, Spain, and the United States' UNOS National Pilot Program for Kidney Paired Donation, adopt similar protocols (Ashlagi and Roth, 2011; Keizer et al., 2005; NHS Blood and Transplant, 2009; Organización Nacional de Transplantes, 2009; Park et al., 2004).

and the better is the match between donors and recipients. One way to enlarge the pool is to offer incentives to compatible patient-donor pairs to enroll in PKE programs. Compatible pairs are not usually considered in PKE program, but they may be very useful for finding additional mutually compatible swaps for incompatible pairs (Gentry et al., 2007; Roth et al., 2005*a*, 2006). To interest them in taking part in such exchanges, however, patients who already have a compatible donor need to be offered the chance of a better match. That is, a graft that can be expected to survive longer than the organ they have been offered by a willing compatible donor. Medical research supports the idea different compatible kidneys can have substantially different outcomes in terms of the expected survival of the graft. In fact, the age and health status of the donor have a major impact on the expected survival of the graft (Gentry et al., 2007; Gjertson, 2004; Su, Zenios and Chertow, 2004). For instance, recent studies by Øien et al. (2007) confirm that the donor's age and health status have a crucial role in the case of living donations. A donor over 65 years old is associated with a higher risk of graft loss at all time points after transplantation.³

In this paper, we suggest protocols (kidney assignment rules) designed to induce compatible donor-patient pairs to participate in PKE programs by affording patients the chances of being matched with a donor younger than their own compatible donors, and thus receive a kidney with a predictably longer survival. The existence of such well-behaved rules is far from being obvious if patients are not assumed to be indifferent between any pair of compatible kidneys.⁴ Since, the donors' age and state of health have the same influence on all patients, this fact calls for a natural restriction on patients' preferences over kidneys. In this restricted preference domain we can design rules that satisfy strong normative requirements and take the different quality of the available kidneys into

³There is some controversy in the medical literature regarding the effects on graft survival of other medical features, e. g. similarity of tissue types between patients and donors (Delmonico, 2004; Gjertson and Cecka, 2000; Opelz, 1997; Opelz for the Collaborative Transplant Study, 1998).

⁴Under feasibility constraints on the maximum number of exchanges that can be simultaneously performed, if quality concerns regarding the compatible kidneys are taken into account and patients' preferences concerning the available kidneys are not restricted, then any rule meeting minimal efficiency requirements does not induce patients to truthfully reveal their preferences. (Nicoló and Rodríguez-Álvarez, 2009; Roth et al., 2005*b*).

account.

We incorporate the implications of the donors' features on the patients' preferences in a PKE framework. We analyze a model of non-monetary exchange of indivisible goods with initial endowments where patients' preferences are naturally restricted. Patients' preferences over donors' kidneys depend on donor-patient compatibility and on the quality of the kidney. Donor-patient compatibility relates to details that may not be publicly available, being private information for the patients. The quality of donors' kidneys depends on directly observable features as the donor's age and health status, and affects all the potential recipients in much the same manner. The pool of available organs can be divided between equivalence classes of similar quality kidneys. For each pair of kidneys belonging to different equivalence classes, every patient for whom these two kidneys are better than her donor's kidney, ranks those kidneys in the same way. We dub such domains as age-based preferences, given that patients prefer younger compatible donors to older compatible donors.

PKE programs periodically use the donors' and patients' medical details to find compatible exchanges in a pool of donor-patient pairs. This information has to be elicited from patients (or they doctors), and patients may have incentives to provide false information to improve their chances of gaining a better outcome from the process. We therefore model PKE protocols as rules that assign the donors' kidneys to the patients taking patients' preferences over kidneys (or the medical features of patients and donors) into account. We analyze rules defined on the domain of age-based preferences that satisfy *individual rationality*,⁵ *efficiency* restricted by the logistic constraints on the number of donor-patient pairs involved in the exchanges, and *strategy-proofness*.⁶ We first show that those properties are incompatible for rules that allow for simultaneous swaps involving more than two donor-patient pairs. Without loss of generality, we thus focus on rules that only allow for pairwise exchanges. We present a family of rules –*age-based priority rules*–that adapt the real-life protocols to our restricted domain of preferences. According

⁵A rule satisfies *individual rationality* if patients never prefer the initial situation to the assignment prescribed by the rule.

⁶A rule satisfies *strategy-proofness* if patients never have incentives to misrepresent their preferences.

to age-based priority rules, patients select the assignments they prefer from the set of individually rational assignments, and patients' positions in the sequential choice procedure are determined by the quality of their donors. If patients can strictly rank all compatible kidneys, then the age-based priority rule is the unique rule that satisfies *individual rationality*, *2-efficiency*, and *strategy-proofness*. If patients may be indifferent among kidneys, then age-based priority rules still fulfill those properties. In this case, under the additional assumption that a change in a given patient's preferences affects the other patients only if it changes the kidney the patient receives (*non-bossiness*), the class of rules that satisfy the proposed properties is not significantly enlarged with respect to the strict preferences scenario.⁷ Finally, we analyze the relevant extension of the problem in which patients may have more than one potential donor. In this scenario, it seems desirable to investigate rules that never give patients reason to conceal some of their potential donors. We extend the definition of age-based priority rules and fine-tune these rules in order to accomplish this goal.

Our analysis is also relevant to national (centralized) PKE programs where all the relevant medical information on donors and patients is available to the program coordinators. Indeed, compatibility issues are determined by blood-tests carried out directly by the program coordinators (Ashlagi and Roth, 2011; Nicolás and Rodríguez-Álvarez, 2009). There is a crucial item of information about the patients that remains private, however, i.e. the minimum kidney quality demanded by a patient for undergoing transplantation. If patients perceive that the national program may result in them being offered an unacceptable kidney, many donor-patient pairs will be reluctant to enroll. To promote the participation of donor-patient pairs, this information should be elicited from the patients. The problem of incentives thus persists even if the information about compatibility is public. Our main findings are robust to the introduction of public information about donor-patients compatibility. Our focus on pairwise exchanges is still without loss of generality. Since the introduction of additional information reduces the manipulation possibilities for the patient, age-based priority rules retain their desirable properties. Moreover, if patients'

⁷*Non-Bossiness* is specially reasonable in the PKE setting because its violation may lead to monetary transfers (bribes) between patients.

preferences are strict, the characterization result holds in this framework. Also, matching rules that satisfy non-bossiness as well as the other desirable properties, cannot maximize the number of transplants at each preference profile, so centralized PKE protocols that try to maximize the number of kidneys exchanges are still liable to strategic manipulation by the patients.⁸

Our paper contributes to two branches of the matching literature. On the one hand, the literature on kidney exchange, on the other, the literature on strategy-proof allocation of indivisible goods in economies without money when agents have private endowments.

The literature on kidney exchange started with Roth et al. (2004), who show the potential benefits of PKE in terms of increasing the number of kidneys transplanted. This paper assumes that patients' preferences are not restricted and they do not take into account feasibility constraints concerning the minimum number of operations required in the exchange. Alternatively Roth et al. (2005*a,b*) present a theoretical model that supports the priority protocols applied in the NEPKE. These papers explicitly introduce the problem of feasibility constraints and preference domain restrictions in PKE. Basically, they analyze the incentives and equity properties of priority rules when kidneys exchanges are restricted to involve just two donor-patient pairs and it is indifferent to patients which compatible kidneys are given. In this scenario, Roth et al. (2007) show that efficiency gains could be achieved (and almost exhausted) if kidney exchanges between three donor-patient pairs were allowable, assuming that donor-patient compatibility depends only on blood-type. Hatfield (2005) characterizes all the rules that satisfy and *constrained efficiency* and *strategy-proofness* for arbitrary feasibility restrictions. Sönmez and Ünver (2010) further investigate the introduction of compatible donor-patient pairs, and Yilmaz (2011) analyzes egalitarian issues in the NEPKE framework. The main difference between our own and the above papers lies in that the others assumed a dichotomous domain. Ünver

⁸Here, we focus on the simple model where compatibility data are considered confidential information. The extension to the public information about compatibility entails cumbersome notation and the results become context specific (i.e. depending on the compatibility information). The main intuitions and results deriving from the private information framework nonetheless remain valid. A complete discussion of the extended framework is available in the working paper version of this article. (See Appendix B.)

(2010) explicitly incorporates a dynamic analysis of the problem, investigating the design of efficient exchange programs on the assumption that patients consider all compatible kidneys equally valid.⁹ We differ from the above mentioned papers in that we introduce a richer domain of preferences, that admits strict preference over compatible kidneys, and which inevitably narrows the class of rules satisfying our requirements. Finally, Nicoló and Rodríguez-Álvarez (2009) show that these properties are incompatible with unrestricted preferences under feasibility constraints and analyze patients' behavior if they are extremely risk averse. In the present paper, we show that this incompatibility can be avoided if patients preferences are restricted to be age-based.

As for the literature on the strategy-proof allocation of objects, a vast amount of research has been conducted on the agents' incentives in the so-called housing markets proposed by Shapley and Scarf (1974). Shapley and Scarf (1974) show that when agents are endowed with only one object the set of (strict) core assignments (in the sense that no group of agents can gain by swapping objects among themselves) coincides with the competitive equilibrium correspondence. When preferences are strict, the core is single-valued (Roth and Postlewaite, 1977) and the rule that selects the core allocation has many desirable properties. (Bird, 1984; Miyagawa, 2002; Roth, 1982). In fact, it is the only rule that satisfies *individual rationality*, (unconstrained) *Pareto-efficiency*, and *strategy-proofness* (Ma, 1994; Sönmez, 1999). Most of the desirable properties no longer hold, however, when indifferences are allowed and/or there are feasibility constraints. The (strict) core itself may not exist (Shapley and Scarf, 1974), *Pareto efficiency* and *group strategy-proofness* become incompatible (Ehlers, 2002). The recent papers by Alcalde-Unzu and Molis (2010) and Jaramillo and Manjunath (2009) show that there are rules that satisfy *individual rationality*, *Pareto-efficiency*, and *strategy-proofness*, but they never satisfy *anonymity*. We contribute to this literature by analyzing a domain restriction where there are rules that satisfy most of the desirable properties, in the presence of weak preferences and feasibility constraints. Finally, Pápai (2007) studies trading in general

⁹Besides Ünver (2010), Zenios (2002) also considers PKE in a dynamic setting. Zenios focuses on the optimal assignment of donor-patient pairs to direct exchange programs or indirect exchange programs, where patients could swap their incompatible donor to gain priority on the waiting list. In this paper the planner is perfectly aware of the patients' preferences and no information has to be elicited.

markets where individuals are endowed with multiple heterogeneous indivisible goods and a feasible allocation is a reallocation of the indivisible goods among the agents. Given the nature of the problem that we study, we also consider the case where individuals may have multiple donors, but we assume that only one object is exchanged. We also depart from Pápai (2007) in that we assume that agents may have confidential information not only regarding their preferences but also the set of initial objects. We therefore focus on the agents' incentives to truthfully report the information about both their preferences and their initial endowment.

Before going on to the formal analysis, we briefly outline the contents of the remainder of this paper. In Section 2, we present the model of kidney assignment problems and basic notation. In Section 3, we introduce the concept of age-based preferences. In Section 4, we analyze the framework in which each patient may have only one willing donor. In Section 5, we extend the analysis to the case of multiple. In Section 6, we present concluding remarks and further applications of the framework, and we provide all the proofs in the Appendix.

2 Basic Notation

Consider a finite society consisting of a set $N = \{1, \dots, n\}$ of patients ($n \geq 3$) who need a kidney for transplantation. Each patient has a potential donor, and $\Omega = \{\omega_1, \dots, \omega_n\}$ denotes the set of kidneys available for transplantation. For each patient i , ω_i refers to the kidney of patient i 's donor. We assume for the moment that all available kidneys are obtained through living donors and each patient has only one potential donor.¹⁰

Each patient i is equipped with a complete, reflexive, and transitive preference relation \succsim_i on Ω . We denote by \succ_i the associated strict preference relation and by \sim_i the associated indifference relation. Let \mathcal{P} denote the set of all preferences. We call $\succsim \in \mathcal{P}^N$ a preference profile. For each $T \subseteq N$ and each $\succsim \in \mathcal{P}^N$, $\succsim_T \in \mathcal{P}^T$ denotes the restriction of the profile \succsim for the members of T . We usually assume that patients' preferences are further

¹⁰We dispense with the later assumption in Section 5.

restricted, so for each patient i her preferences belong to a subset $\mathcal{D}_i \subset \mathcal{P}$. We denote by $\mathcal{D} \equiv \times_{i \in N} \mathcal{D}_i \subseteq \mathcal{P}^N$ a domain of preferences over kidneys.

An **assignment** is a bijection from kidneys to patients. We denote an arbitrary a as an n -tuple of pairs $a = [(1, \omega), \dots, (n, \omega')]$ such that for each $i, j \in N$, $i \neq j$ and each $\omega, \omega' \in \Omega$, if $(i, \omega), (j, \omega') \in a$, then $\omega \neq \omega'$. For each patient i and each assignment a , a_i denotes the kidney assigned to i by a . For each assignment a , if a patient is assigned her donor's kidney ($a_i = \omega_i$), we interpret that either she continues in dialysis or –if she is compatible with her donor– she receives her donor's kidney. Let \mathcal{A} be the set of all assignments.

In every assignment, kidneys are allocated by forming exchange cycles of patient–donors couples. In each cycle, every patient receives a kidney from the donor of some patient in the cycle and simultaneously her donor's kidney is transplanted to another patient in the cycle.

For each assignment a , let π_a be the finest partition of the set of patients such that for each $p \in \pi_a$ and each $i \in p$, there are $j, j' \in p$, with $a_i = \omega_j$ and $a_{j'} = \omega_i$.¹¹

Clearly, for each assignment a the partition π_a is unique and well-defined. We define the **cardinality of a** as the $\max_{p \in \pi_a} \#p$.

The cardinality of an assignment refers to the size of the largest cycle formed in the assignment. The concept of cardinality is crucial for our notion of feasibility.

For each $k \in \mathbb{N}$, $k \leq n$, we say that the assignment a is **k -feasible** if a 's cardinality is not larger than k . Let \mathcal{A}^k be the set of all k -feasible assignments.

An interesting case of feasibility restrictions appears when only immediate exchanges between two donor-patient pairs are admitted. An assignment a is a **pairwise-exchange** assignment ($a \in \mathcal{A}^2$) if a satisfies that if for some $i, j \in N$ $(i, \omega_j) \in a$, then $(j, \omega_i) \in a$.

In this paper, we are interested in rules that select a (kidney) assignment for each preference profile. A rule defined in the domain \mathcal{D} is a mapping $\varphi : \mathcal{D} \rightarrow \mathcal{A}$.

¹¹Note that $j = j'$ and $i = j = j'$ and then $a_i = \omega_i$ are allowed.

Finally, we present formal definition of the standard desirable conditions for rules.

Individual Rationality. For each $i \in N$ and each $\succsim \in \mathcal{D}$, $\varphi_i(\succsim) \succsim_i \omega_i$.

k -Efficiency. For each $\succsim \in \mathcal{D}$, $\varphi(\succsim) \in \mathcal{A}^k$ and there is no $a \in \mathcal{A}^k$ such that for each $i \in N$ $a_i \succsim_i \varphi_i(\succsim)$ and for some $j \in N$, $a_j \succ_j \varphi_j(\succsim)$.

Strategy-Proofness. For each $i \in N$, each $\succsim \in \mathcal{D}$, and each $\succsim'_i \in \mathcal{D}_i$,

$$\varphi_i(\succsim) \succsim_i \varphi_i(\succsim'_i, \succsim_{-i}).$$

3 Age-Based Preferences

In this section we present a new domain restriction that is directly inspired by the structure of paired kidney exchanges. Before describing the domain restriction, we introduce some useful notation.

For each patient i and each preference $\succsim_i \in \mathcal{P}$, we define i 's **set of desirable kidneys** $D(\succsim_i) \equiv \{\omega \in \Omega \mid \omega \succ_i \omega_i\}$, and analogously i 's **set of undesirable kidneys** $ND(\succsim_i) \equiv \{\omega \in \Omega \setminus \{\omega_i\} \mid \omega_i \succsim_i \omega\}$.

We interpret each patient's set of desirable kidneys as all those kidneys which lead to an improvement with respect to the patient best outside option, that is either to staying in dialysis or to receiving her donor's organ. Note that a patient with a compatible donor may prefer to remain under dialysis if the transplantation does not lead to a significant increase in her life expectancy.

Kidney transplantations from living donors have excellent long-term outcomes irrespective of matching according to HLA type (Delmonico, 2004; Gjertson and Cecka, 2000). Donor's characteristics, like age or health status, however have a significant impact on graft survival. We incorporate this fact by assuming that patients divide the set of desirable kidney in subsets of kidneys of homogeneous quality (indifference classes). Patients are not always indifferent between pairs of compatible kidneys. Specifically, the quality

of the match depends on characteristics of the donors which are observable by the transplant coordinator and *these characteristics affect all patients in the same direction*. In fact, donor's age turns out to represent the most relevant characteristic to determine the probability of long-term graft survival in case of living donations.

Let denote by $\Pi = \{\Pi(1), \dots, \Pi(l)\}$ be a partition of Ω . For each patient $i \in N$, the preference relation $\succsim_i \in \mathcal{P}$ is a **Π -based preference** if for each $\omega, \omega' \in D(\succsim_i)$ and for each $\bar{\omega}, \bar{\omega}' \in ND(\succsim_i)$:

- (i) $\omega \in \Pi(j)$ and $\omega' \in \Pi(k)$ and $j < k$ imply $\omega \succ_i \omega'$, and
- (ii) $\omega, \omega' \in \Pi(j)$ implies $\omega \sim_i \omega'$.
- (iii) $\omega_i \succ_i \bar{\omega}$ and $\bar{\omega} \sim_i \bar{\omega}'$.

Let \mathcal{D}_i^Π denote the set of all Π -based preferences for patient i and let $\mathcal{D}^\Pi \equiv \times_{i \in N} \mathcal{D}_i^\Pi$.

According to Π -based preferences, Π divides (in decreasing order) the set of available kidneys in subsets of homogeneous quality kidneys. For expositional clarity, from now on we assume that Π partitions the set of available organs for transplantation according to the donors' age. We call the induced preference domain \mathcal{D}^Π the Π -age-based preference domain. Thus, $\Pi(1)$ contains the highest quality kidneys, $\Pi(2)$ the second highest quality kidneys, and $\Pi(l)$ the lowest quality kidneys. In the simplest setting, the set of available kidneys is divided in two disjoint subsets, young donors and mature donors. This partition induces a natural restriction on patients' preferences. Whether being a young or a mature donor kidney does not determine the compatibility between the donor and the patient, a desirable kidney from a young donor is preferred to a desirable kidney from a mature donor. Moreover, patients are indifferent between any pair of desirable kidney from young (mature) donors.

We assume that the partition Π is public information. Patients report which is the set of desirable kidneys. It is worth highlighting that since we focus on rules that satisfy *individual rationality*, all the relevant information that is private for patient i is contained in her sets of desirable and undesirable kidneys. For each $\succsim_i \in \mathcal{D}^\Pi$, if both $\omega, \omega' \in D(\succsim_i)$,

the order in which i ranks ω with respect to ω' only depends on the elements of the partition Π they belong to.

Throughout the paper we rule out degenerate situations where the set of kidneys is partitioned in two sets and one of them contains a unique element.

Assumption. The partition of the set of kidneys Π is such that either $\#\Pi \geq 3$ or $\Pi = \{\Pi(1), \Pi(2)\}$ and $\#\Pi(1) \geq 2$ and $\#\Pi(2) \geq 2$.

Without any loss of generality and in order to simplify notation, henceforth, we assume that for each partition Π , for each $i, j \in N$, with $\omega_i \in \Pi(l)$ and $\omega_j \in \Pi(l')$, $i < j$ implies $\Pi(l) \leq \Pi(l')$.

4 Priority Rules and Age-Based Preferences

In this restricted age-based domain, we show that positive results may emerge and we are able to present rules that satisfy *individual rationality*, *k-efficiency*, and *strategy-proofness*. Since donor's characteristics have a common effect on patients' preferences, we can escape impossibility results. We start this section, however, with an impossibility result that highlights the tension between *k-efficiency* and *strategy-proofness*. Even in the restricted age-based environment, we have to focus on rules that only allow for pairwise exchanges.

Theorem 1. *For each partition Π and each $k \in \mathbb{N}$ such that $3 \leq k \leq n - 1$, no rule $\varphi : \mathcal{D}^\Pi \rightarrow \mathcal{A}^k$ satisfies individual rationality, k -efficiency, and strategy-proofness.*

The literature on kidney exchanges has focused on priority mechanisms which are commonly used in most transplant centers to allocate cadaveric organs (Roth et al., 2005a,b). In this section, we analyze how priority mechanisms need to be tailored in the age-based preferences environment.

A **priority ordering** σ is a permutation of patients ($\sigma : N \rightarrow N$) such that the k -th patient in the permutation is the patient with the k -th priority. Let σ^* denote the natural

priority ordering (for each $i \in N$, $\sigma^*(i) = i$). For each partition Π and each priority ordering σ , we say that σ **respects** Π if for each $i, j \in N$ with $\omega_i \in \Pi(l)$, $\omega_j \in \Pi(l')$, $l < l'$ implies $\sigma(i) < \sigma(j)$.

For each $\succsim \in \mathcal{P}^N$, and each $l \leq n$, let $\mathcal{I}(\succsim) \equiv \{a \in \mathcal{A}^2 \mid \text{for each } i \in N \ a_i \succsim_i \omega_i\}$ denote the **set of all individually rational pairwise assignments**.

Priority Algorithm. Fix a permutation of the patients σ , and a preference profile $\succsim \in \mathcal{P}^N$:

- Let $\mathcal{M}_0^\sigma(\succsim) = \mathcal{I}(\succsim)$.
- For each $k \leq n$, let $\mathcal{M}_k^\sigma \subseteq \mathcal{M}_{k-1}^\sigma$ be such that:

$$\mathcal{M}_k^\sigma(\succsim) = \{a \in \mathcal{M}_{k-1}^\sigma \mid \text{for no } b \in \mathcal{M}_{k-1}^\sigma(\succsim), b_{\sigma^{-1}(k)} \succ_{\sigma^{-1}(k)} a_{\sigma^{-1}(k)}\}.$$

Note that \mathcal{M}_n^σ is well defined, non-empty, and essentially single-valued.¹²

Let \mathcal{D} be an arbitrary domain of preferences. A rule $\varphi : \mathcal{D} \rightarrow \mathcal{A}^2$ is a **pairwise priority rule** if there is a priority ordering σ such that for each $\succsim \in \mathcal{D}$, $\varphi(\succsim) \in \mathcal{M}_n^\sigma(\succsim)$. We denote by ψ^σ the pairwise priority rule with priority ordering σ . Analogously, let $\psi^* \equiv \psi^{\sigma^*}$. Finally, for every partition Π and priority ordering σ , the pairwise priority rule ψ^σ is an **age-based priority rule** if σ respects Π .¹³

At this point, we introduce the concept of the core in PKE problems.

For each pair of pairwise assignments $a, b \in \mathcal{A}^2$, each coalition $T \subseteq N$, and each $\succsim \in \mathcal{P}^N$, a **weakly dominates** b via T at \succsim if:

- (i) For each $i \in T$, there is $j \in T$ such that $a_i = \omega_j$,

¹²A set is essentially single-valued if either it is single-valued, or if it contains more than one element, all the patients are indifferent between any two elements in the set. That is, for each patient i , each $\succsim \in \mathcal{P}^N$, and each $a, a' \in \mathcal{M}_n^\sigma(\succsim)$, $a_i \sim_i a'_i$.

¹³Note that different priority orderings may generate the same **priority rule**. In fact, when the last element of the partition Π is a singleton, there are priority orderings that do not respect Π that generate an **age-based priority rule**. Cfr. Remarks 2 and Lemma 2 in the Appendix.

- (ii) for each $i \in T$, $a_i \succsim_i b_i$,
- (iii) there is $j \in T$, such that $a_j \succ_j b_j$.

In this case we say that coalition T **weakly blocks** b under \succsim via a . For each $\succsim \in \mathcal{P}^N$, an assignment $a \in \mathcal{A}^2$ is in the **strict core** of the pairwise exchange problem associated to \succsim if a is not weakly dominated by any assignment $b \in \mathcal{A}^2$. Similarly, a **strongly dominates** b via T at \succsim if:

- (i) For each $i \in T$, there is $j \in T$ such that $a_i = \omega_j$,
- (ii) for each $i \in T$, $a_i \succ_i b_i$.

An assignment $a \in \mathcal{A}^2$ is in the **weak core** of the pairwise exchange problem associated to \succsim if a is not strongly dominated by any assignment $b \in \mathcal{A}^2$.¹⁴

We first consider situations where patients are never indifferent between two acceptable kidneys. Let Π^* denote the complete partition according to the natural order, that is, $\Pi^* \equiv \{\Pi(1), \dots, \Pi(n)\} = \{\{1\}, \dots, \{n\}\}$. When the set of available kidneys is finely partitioned and each indifference class consists of a single donor, the partition induces strict preferences over the set of available kidneys. Note that since preferences are strict, for each $\succsim \in \mathcal{D}^{\Pi^*}$, $\mathcal{M}_n^{\sigma^*}(\succsim)$ is always single-valued. In our next result, we state the close relation between age-based priority rules and the strict core in the domain \mathcal{D}^{Π^*} .

Proposition 1. *For each $\succsim \in \mathcal{D}^{\Pi^*}$, $\psi^*(\succsim)$ is the unique assignment in the strict core of the pairwise exchange problem associated to \succsim .*

With Proposition 1 at hand, we characterize the age-based priority rule as the unique rule that satisfies *individual rationality*, *pairwise k -efficiency*, and *strategy-proofness* in \mathcal{D}^{Π^*} .

Theorem 2. *A rule $\varphi : \mathcal{D}^{\Pi^*} \rightarrow \mathcal{A}^2$ satisfies individual rationality, 2-efficiency and strategy-proofness if and only if φ is the age-based priority rule ψ^* .*

¹⁴Note that the notions of strict and weak core coincide if preferences are strict.

The previous result does not extend to coarser partitions of the set of available kidneys. For arbitrary partitions Π , there are many profiles of age based preferences such that the strict core is empty. The arguments in the proof of Theorem 2 only apply to domains that always generate non-empty strict cores (See Sönmez (1999)). This fact notwithstanding, with the arguments in the proof of Proposition 1, it is immediate to check that every age-based priority rule selects an assignment in the weak core. We devote the rest of this section to highlight the relevance of age-based priorities in the general framework by analyzing the implications of additional properties.

Group Strategy-Proofness. For each $\succsim \in \mathcal{D}$, there is no $T \subseteq N$ and $\succsim'_T \in \mathcal{D}_T$ such that for each $i \in T$, $\varphi_i(\succsim'_T, \succsim_{N \setminus T}) \succ_i \varphi_i(\succsim)$.

Group strategy-proofness requires that for each patient truthfully reporting her preferences is a dominant strategy and coalitions of patients never have incentives to coordinate to jointly misreport their preferences. Note that our definition of *group strategy-proofness* only considers situations in which all the members of a deviating coalition strictly improve upon the initial report.¹⁵ *Group strategy-proofness* is a relevant property for kidney assignment rules for the following reason. Patients belonging to the same transplant unit often know each others very well and have strong emotional relations among them, which makes quite reasonable to assume that they could and want to jointly misreport their preferences if this could benefit all of them. Moreover often patients' preferences are reported by their doctors to the transplant coordinators. Doctors can easily help the exchange of information and the coordination among their patients.

If patients may be indifferent between desirable kidneys, then only *age-based priority rules* satisfy *strategy-proofness* among all *priority rules*. Furthermore, *age-based priority rules* do not provide incentives for any group of agents to misreport their preferences.

Theorem 3. For each partition Π and each priority ordering σ , the priority rule ψ^σ satisfies *strategy-proofness* in \mathcal{D}^Π if and only if ψ^σ is an *age-based priority rule*. Moreover,

¹⁵A stronger definition of *group strategy-proofness* that allows for that some members of the coalition to remain indifferent would be incompatible with the remaining properties in our framework. See Ehlers (2002).

if σ respects Π , then the age-based priority rule ψ^σ satisfies group strategy-proofness in \mathcal{D}^Π .

Theorem 3 shows the relevance of *age-based priority rules* in the class of *priority rules* and their nice properties in terms of group incentives. We conclude this section providing further evidence on the central position of *age-based priority rules* among the rules that satisfy *individual rationality*, *2-efficiency*, and *strategy-proofness* in the domain of age-based preferences with indifferences. To perform this task, we consider the following property.

Non-Bossiness. For each $i \in N$, each $\succsim \in \mathcal{D}$, and each $\succsim'_i \in \mathcal{D}_i$, $\varphi_i(\succsim) = \varphi_i(\succsim'_i, \succsim_{-i})$ implies $\varphi(\succsim) = \varphi(\succsim'_i, \succsim_{-i})$

Non-Bossiness requires that if any patient i gets the same kidney under two preference profiles which differ only for patient i 's preferences, then all patients get the same kidney under the two profiles. In fact, a rule that violates *non-bossiness* could be prone to (illegal) bribes among donor-patient pairs. If a patient i changes her report of preferences and she affects the outcome of patient j , then i may have incentives to accept any monetary compensation from j to reverse her report. This property seems particularly interesting in our setting where only pairwise exchanges are admissible and monetary transactions are prohibited.

It is clear also that *age-based priority rules* satisfy *non-bossiness*. Moreover, *non-bossiness* in combination with our initial properties has interesting implications. First, this set of axioms precludes the maximization of the number of mutually compatible swaps.

Lemma 1. *If $\varphi : \mathcal{D}^\Pi \rightarrow \mathcal{A}^2$ satisfies individual rationality, 2-efficiency, strategy-proofness, and non-bossiness then for each $i, j \in N$ and $\succsim \in \mathcal{D}^\Pi$, $\omega_i \in D(\succsim_j)$ and $\omega_j \in D(\succsim_i)$ imply either $\varphi_i(\succsim) \succsim_i \omega_j$ or $\varphi_j(\succsim) \succsim_j \omega_i$ (or both).*

It is worth noticing that the implication of Lemma 1 holds if *strategy-proofness* and *non-bossiness* are replaced with *group strategy-proofness*.

Example 1. Let $N = \{1, 2, 3, 4\}$ and $\Pi(1) = \{\omega_1, \omega_2\}$, $\Pi(2) = \{\omega_3, \omega_4\}$. Consider the preference profile $\succsim \in \mathcal{D}^\Pi$ such that

\succsim_1	\succsim_2	\succsim_3	\succsim_4
ω_2	ω_1	ω_1	ω_2
$\omega_3 \sim_1 \omega_4$	$\omega_3 \sim_1 \omega_4$	ω_3	ω_4
ω_1	ω_2	\dots	\dots

Note that $[(1, \omega_3), (2, \omega_4), (3, \omega_1), (4, \omega_2)] \in \mathcal{I}(\succsim)$. For every rule φ that satisfies individual rationality, 2-efficiency, strategy-proofness, and non-bossiness, however,

$$\varphi(\succsim) = [(1, \omega_2), (2, \omega_1), (3, \omega_3), (4, \omega_4)].$$

Adding *non-bossiness* to the set of desirable properties of the matching rule has a second important consequence. In fact these four properties call for rules that always pick assignments in the weak core of the associated pairwise exchange problem.

Proposition 2. If $\varphi : \mathcal{D}^\Pi \rightarrow \mathcal{A}^2$ satisfies individual rationality, 2-efficiency, strategy-proofness, and non-bossiness then for each $\succsim \in \mathcal{D}^\Pi$, $\varphi(\succsim)$ selects an assignment in the weak core of the pairwise exchange problem associated to \succsim .

While a full characterization of the rules that satisfy *individual rationality*, *2-efficiency*, *strategy-proofness*, and *non-bossiness*, seems to be out of reach, we can take a further important step in understanding the structure of such rules. In order to simplify the exposition, we conclude this section analyzing the case in which there are only two types of donors, young and mature donors. Thus, $\Pi = \{\Pi(1), \Pi(2)\}$. In this simple scenario, we can provide a clear description of the implications of *individual rationality*, *2-efficiency*, *strategy-proofness*, and *non-bossiness*. The results presented here can be immediately extended to the general framework with arbitrary partitions of the set of available kidneys. Since the extension calls for additional notations, we leave it for the interested reader and it is relegated to the Appendix.¹⁶

¹⁶See Theorem 6.

For each $a \in \mathcal{A}^2$, define:

$$\begin{aligned} M_{1,1}(a) &\equiv \{i \in N \mid a_i \neq \omega_i \ \& \ \omega_i \in \Pi(1) \ \text{and} \ a_i \in \Pi(1)\}, \\ M_{1,2}(a) &\equiv \left\{ i \in N \mid a_i \neq \omega_i \ \& \ \begin{array}{l} \text{either } \omega_i \in \Pi(1) \ \text{and} \ a_i \in \Pi(2) \\ \text{or } \omega_i \in \Pi(2) \ \text{and} \ a_i \in \Pi(1) \end{array} \right\}, \\ M_{2,2}(a) &\equiv \{i \in N \mid a_i \neq \omega_i \ \& \ \omega_i \in \Pi(2) \ \text{and} \ a_i \in \Pi(2)\}. \end{aligned}$$

A rule φ is a *sequential matching maximizing rule* if for each $\succsim \in \mathcal{D}^\Pi$:

- i) $\#M_{1,1}(\varphi(\succsim)) \geq \#M_{1,1}(a)$, for each $a \in \mathcal{I}(\succsim)$,
- ii) $\#M_{1,2}(\varphi(\succsim)) \geq \#M_{1,2}(a')$, for each $a' \in \mathcal{I}(\succsim)$ such that $a'_i = \varphi_i(\succsim)$ for each $i \in M_{1,1}(\varphi(\succsim))$,
- iii) $\#M_{2,2}(\varphi(\succsim)) \geq \#M_{2,2}(a'')$, for each $a'' \in \mathcal{I}(\succsim)$ such that $a''_i = \varphi_i(\succsim)$ for each $i \in [M_{1,1}(\varphi(\succsim)) \cup M_{1,2}(\varphi(\succsim))]$.

A *sequential matching maximizing rule* does not necessarily select an assignment that maximizes the number of compatible kidneys exchanges. Instead, it chooses an assignment that maximizes the number of swaps among pairs of patients with a young donor. Then, it sequentially applies the same logic to the remaining donor-patient pairs. Thus, given the exchanges of among pairs of patients with a young donor, it maximizes the number of swaps among pairs of patients such that only one patient has a young donor. Finally, given the exchanges arranged in the previous stages, a *sequential matching maximizing rule* maximizes the number of swaps between the remaining pairs. Note that *age-based priority rules* are *sequential matching maximizing rules*.

Theorem 4. *If $\varphi : \mathcal{D}^\Pi \rightarrow \mathcal{A}^2$ satisfies individual rationality, 2-efficiency, strategy-proofness, and non-bossiness, then φ is a sequential matching maximizing rule.*

It is worth noticing that Theorem 4 implies that there may be assignments in the weak core that are never selected by a rule that satisfies our set of axioms. Consider $N = \{1, 2, 3, 4, 5, 6\}$ and $\Pi = \{\Pi(1), \Pi(2)\} = \{\{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6\}\}$. Let $\succsim \in \mathcal{D}^\Pi$ be such that $D(\succsim_1) = \{\omega_2, \omega_3\}$, $D(\succsim_2) = \{\omega_1, \omega_4\}$, $D(\succsim_3) = \{\omega_1, \omega_5\}$, $D(\succsim_4) = \{\omega_2, \omega_6\}$,

$D(\succsim_5) = \{\omega_3\}$, and $D(\succsim_6) = \{\omega_4\}$. Let $a = ((1, \omega_2), (2, \omega_1), (3, \omega_5), (4, \omega_6), (5, \omega_3), (6, \omega_4))$. Clearly, a is in the weak core of the pairwise exchange problem. It involves three compatible swaps, but only one among patients with a donor in $\Pi(1)$. A rule that satisfies our axioms cannot choose a at profile \succsim because it is possible to carry out two exchanges involving donor-patients pairs in $\Pi(1)$.¹⁷

5 Multiple Donors

Sometimes patients in the waiting list may find more than one potential donors. If patients with multiple potential donors are keen to participate to PKE programs, algorithms have to take account of this aspect. Even if only one among the potential donors of a patient donates her kidney, the fact that a patient may have many potential donors can greatly increase the chances to find mutually compatible pairs. Since it is reasonable to assume that the information about how many potential donors a patient have is private information, rules should provide incentives to patients to reveal this valuable information. To analyze this general case, we need to slightly modify the framework and to incorporate some additional notation.

Let $N = \{i, \dots, n\}$ be a set of patients and $\Omega = \{\omega_1, \dots, \omega_{n'}\}$ be a set of available kidneys from living donors, $n \leq n'$.¹⁸ For each patient i let Ω_i denote the set of kidneys from i 's donors. Clearly, $\cup_{i \in N} \Omega_i = \Omega$ and for each patient $j \neq i$, $\Omega_i \cap \Omega_j = \emptyset$.

In the multiple donor case, we analyze patients' incentives to manipulate by reporting different sets of potential donors. The set of donors is an argument of the kidney assignment rule. For each patient i let \mathcal{K}_i be the set of non-empty subsets of Ω_i and $K_i \in \mathcal{K}_i$ is a set of donors reported by patient i to the transplant coordinators. Let $\mathcal{K} \equiv \times_{i \in N} \mathcal{K}_i$. We denote by $\mathbf{K} = (K_1, \dots, K_n)$ a generic element of \mathcal{K} , let $\mathbf{\Omega} = (\Omega_1, \dots, \Omega_n)$. For each $\mathbf{K} \in \mathcal{K}$ and for each $S \subseteq N$, \mathbf{K}_S denotes the restriction of \mathbf{K} to the members of S .

A **(generalized) assignment** a is an n -tuple of pairs $a = [(1, \omega), \dots, (n, \omega')]$ such that

¹⁷For instance, consider the assignment $b = [(1, \omega_3), (2, \omega_4), (3, \omega_1), (4, \omega_2), (5, \omega_5), (6, \omega_6)]$.

¹⁸Note that kidneys' indexes do not longer refer to the patient who introduces the donor in the pool.

- (i) for each $i, j \in N$, $i \neq j$ and each $\omega, \omega' \in \Omega$, if $(i, \omega), (j, \omega') \in a$, then $\omega \neq \omega'$;
- (ii) for each $i \in N$, if for some $j \in N$, $a_j \in \Omega_i$, then $a_k \notin \Omega_i$ for all $k \neq j$.

We need to introduce this second requirement to convey the idea that for each patient at most one donor donates her kidney.

For each $\mathbf{K} \in \mathcal{K}$, we say that an assignment a is **\mathbf{K} -feasible** if for each $i, j \in N$, $a_i \in \Omega_j$ implies $a_i \in \mathbf{K}_j$. Let $\mathcal{A}(\mathbf{K})$ be the set of all **\mathbf{K} -feasible** (generalized) assignments, and for each $k \leq n$ let $\mathcal{A}^k(\mathbf{K})$ be the set of all **\mathbf{K} -feasible** (generalized) assignments with cardinality smaller or equal to k .

A **generalized (kidney assignment) rule** is a mapping $\Phi : \mathcal{D} \times \mathcal{K} \rightarrow \mathcal{A}$ such that for each $\succsim \in \mathcal{D}$ and each $\mathbf{K} \in \mathcal{K}$, $\Phi(\succsim, \mathbf{K}) \in \mathcal{A}(\mathbf{K})$.

The definition of *k-efficiency* directly applies to the multiple donor scenario. *Individual rationality* may be immediately extended just by applying its logic to all the potential donors of each patient.

Individual Rationality for generalized rules. For each $i \in N$, each $\succsim \in \mathcal{D}$, each $\mathbf{K} \in \mathcal{K}$, and each $\omega \in \mathbf{K}_i$, $\Phi_i(\succsim, \mathbf{K}) \succsim_i \omega$.

The extension of the notion of *strategy-proofness* is more delicate. In the general framework, it is necessary to take into account patients' incentives to manipulate the PKE outcome by withholding their potential donors.

Extended Group Strategy-Proofness (EGSP). There are no $T \subseteq N$, $\succsim \in \mathcal{D}$, $\succsim'_T \in \mathcal{D}_T$, and $\mathbf{K}'_T \in \times_{i \in T} \mathcal{K}_i$, such that for each $i \in T$, $\Phi_i(\succsim'_T, \succsim_{N \setminus T}, (\mathbf{K}'_T, \Omega_{N \setminus T})) \succsim_i \Phi_i(\succsim, \Omega)$.

In the multiple donors scenario, we assume that patients only care about the kidney they may receive and do not have preferences over whom be the willing donor involved in the kidney exchange. Patients' preferences are still defined over all potential donors Ω . The assumption on the irrelevance of the donor for patients' preferences allows to focus on the case in which patients could misreport the set of their potential donors

in order to get a better kidney. To conclude with the description of patients' preferences, we say that a kidney is desirable for patient i if it improves upon all i 's potential donors' kidneys. That is, for each patient i , i 's **set of desirable kidneys** is the set $D(\succsim_i) \equiv \{\omega \in \Omega \mid \forall \omega' \in \Omega_i, \omega \succ_i \omega'\}$. With this definition of desirable kidneys, for each partition $\Pi = \{\Pi(1), \dots, \Pi(l)\}$ of Ω , the notion of age-based preferences in the multiple donor scenario simply replicates the definition in the single-donor case.

The positive results in the previous section directly apply to the multiple donor framework. We thus focus on pairwise exchanges and *age-based priority rules*. Since patients may have donors whose kidneys belong to different classes of the partition Π , there are alternative extensions of the notion of *age-based priority rules* to the multiple donor case. We devote this section to show the difficulties that arise in this setting and how to tailor age-based priority rules in order to preserve the properties they satisfy in the single-donor scenario.

Example 2. Let $N = \{1, 2, 3\}$, $\Omega = \{\omega_1, \dots, \omega_6\}$, $\Pi = \{\Pi(1), \Pi(2)\}$ such that $\Pi(1) = \{\omega_1, \omega_2\}$, $\Pi(2) = \{\omega_3, \omega_4, \omega_5, \omega_6\}$ and $\Omega_1 = \{\omega_1, \omega_4\}$, $\Omega_2 = \{\omega_2, \omega_5\}$ and $\Omega_3 = \{\omega_3, \omega_6\}$. Let the generalized rule Φ be defined in such a way that, for each $\mathbf{K} \in \mathcal{K}$, $\Phi(\cdot, \mathbf{K})$ is a priority rule with priority ordering σ^* . The rule Φ assigns priority to patients with a donor in $\Pi(1)$. Let $\succsim \in \mathcal{D}^\Pi$ be such that $D(\succsim_1) = \{\omega_3\}$, $D(\succsim_2) = \{\omega_6\}$ and $D(\succsim_3) = \{\omega_2, \omega_4\}$. Let \mathbf{K}, \mathbf{K}' be such that $\mathbf{K} = \Omega$, $K'_j = \Omega_j$ for $j \in \{1, 2\}$ and $K'_3 = \{\omega_6\}$. Note that

$$\Phi(\succsim, \mathbf{K}) = [(1, \omega_3), (2, \omega_2), (3, \omega_4)],$$

but

$$\Phi(\succsim, \mathbf{K}') = [(1, \omega_1), (2, \omega_6), (3, \omega_2)].$$

Hence, $\Phi_3(\succsim, \mathbf{K}') \succ_3 \Phi_3(\succsim, \mathbf{K})$, and Φ violates EGSP. Moreover, patient 3 could obtain the same outcome just by reporting $\succsim'_3 \in \mathcal{D}_3^\Pi$ such that $D(\succsim'_3) = \{\omega_2\}$.

With multiple donors, it is necessary to define a multi-stage mechanism in order to maintain the non-manipulability of the age-based priority rule. When potential kidneys are divided in young and mature kidneys, Theorem 3 suggests that patients with young donors have priority over patients with mature donors. It is not immediate, however, to

assign the priorities when patients may simultaneously have young and mature donors. To preserve *strategy-proofness*, it is necessary that younger kidneys are offered first.

Let $S \subseteq N$. A **reduced assignment** a_S is an assignment among the members of S . For a partition of N , $\{S, S', \dots, S''\}$ and a list of reduced assignments $a_S, a_{S'}, \dots, a_{S''}$, we abuse notation and refer to the assignment formed by the reduced assignments as $[a_S, a_{S'}, \dots, a_{S''}]$. For each preference profile $\succsim \in \mathcal{P}$ and each profile of available kidneys $\mathbf{K} \in \mathcal{K}$, let $\mathcal{I}(\succsim, \mathbf{K})$ denote the set of all individually rational pairwise assignments when the set of available kidneys is given by \mathbf{K} . For each coalition of patients S , we denote by $\mathcal{I}(\succsim_S, \mathbf{K}_S \mid S)$ the set of individually rational reduced assignments for S under \succsim and \mathbf{K} . Finally for each $T \subset N$ and each permutation of the patients σ , we say that the permutation $\bar{\sigma} : T \rightarrow \{1, \dots, T\}$ is the reduction of σ to T if for each $i, j \in T$, $\sigma(i) < \sigma(j)$ implies $\bar{\sigma}(i) < \bar{\sigma}(j)$.

Let $\Pi = \{\Pi(1), \dots, \Pi(l)\}$ be a partition of Ω . Fix $\succsim \in \mathcal{D}^\Pi$ and $\mathbf{K} \in \mathcal{K}$. Let $\Sigma = \{\sigma_1, \dots, \sigma_l\}$ be a list of permutations of the patients. For each preference profile \succsim , the generalized priority rule generated by the list of permutations Σ proceeds according to the following algorithm:

Let $N(1) \equiv N$ and

$$\mathcal{M}_0^{\Sigma, 1} \equiv \left\{ \begin{array}{l} a \in \mathcal{I}(\succsim, \mathbf{K} \mid N(1)) \text{ such that} \\ \forall i, j \in N(1), \text{ if } a_i \in \Omega_j \text{ and } \sigma_1(i) < \sigma_1(j), \\ \text{then } a_j \in \Pi(1) \end{array} \right\}.$$

For each $k = 1, \dots, \#N(1)$,

$$\mathcal{M}_k^{\Sigma, 1} \equiv \left\{ a \in \mathcal{M}_{k-1}^{\Sigma, 1} \mid \text{for no } b \in \mathcal{M}_{k-1}^{\Sigma, 1}, b_{\sigma_1^{-1}(k)} \succ_{\sigma_1^{-1}(k)} a_{\sigma_1^{-1}(k)} \right\}.$$

Note that $\mathcal{M}_{\#N(1)}^{\Sigma, 1}$ is not empty and essentially single-valued.

Intuitively, according to the priority mechanism, the patients with higher priority offer to swap their donors' kidneys in $\Pi(1)$ to the other patients (maybe receiving a kidney not in $\Pi(1)$.) By the definition of $\mathcal{M}_{\#N(1)}^{\Sigma, 1}$ and the fact that it is essentially single-valued, for every $a \in \mathcal{M}_{\#N(1)}^{\Sigma, 1}$, the set of patients who do not receive a kidney from other patients' donors is the same. In the following stages, the algorithm proceeds by iteratively applying

the same logic to the unmatched patients who are allowed to sequentially offer kidneys in the remaining elements of the partition Π .

Once $N(1)$, $\mathcal{M}_0^{\Sigma,1}$, and $\mathcal{M}_{\#N(1)}^{\Sigma,1}$ are defined, for each $m = 2, \dots, l$; let $N(m) = \{i \in N(m-1) \mid \forall a \in \mathcal{M}_{\#N(m-1)}^{\Sigma,(m-1)}, a_i \in \Omega_i\}$, $M(m-1) = N(m-1) \setminus N(m)$, let $\bar{\sigma}_m$ be the restriction of σ_m to $N(m)$, and let

$$\mathcal{M}_0^{\Sigma,m} \equiv \left\{ \begin{array}{l} a_{N(M)} \in \mathcal{I}(\succ_{N(m)}, \mathbf{K}_{N(m)} \mid N(m)) \text{ such that} \\ \forall i, j \in N(m), \text{ if } a_i \in \Omega_j \text{ and } \bar{\sigma}_m(i) < \bar{\sigma}_m(j), \\ \text{then } a_j \in \Pi(m) \end{array} \right\}.$$

For each $k = 1, \dots, \#N(m)$,

$$\mathcal{M}_k^{\Sigma,m} \equiv \left\{ a_{N(m)} \in \mathcal{M}_{k-1}^{\Sigma,m} \mid \text{for no } b \in \mathcal{M}_{k-1}^{\Sigma,m}, b_{\bar{\sigma}_m^{-1}(k)} \succ_{\bar{\sigma}_m^{-1}(k)} a_{\bar{\sigma}_m^{-1}(k)} \right\}.$$

A (generalized) rule $\Phi : \mathcal{D}^\Pi \times \mathcal{K} \rightarrow \mathcal{A}^2$ is a **generalized priority rule** if there is a list of permutations $\Sigma = \{\sigma_1, \dots, \sigma_l\}$ such that for each $\succ \in \mathcal{D}^\Pi$, for each $m = 1, \dots, l$, for each $a^m \in \mathcal{M}_{N(m)}^{\Sigma,m}$, $\Phi(\succ) = [\Phi(\succ)_{M(1)}, \dots, \Phi(\succ)_{M(l)}]$ and $\Phi(\succ)_{M(m)} = a_{M(m)}^m$. For a partition Π , $\Psi : \mathcal{D}^\Pi \times \mathcal{K} \rightarrow \mathcal{A}^2$ is a **generalized age-based priority rule** if Ψ is a *generalized priority rule* and for each $i, j \in N$, for each $m = 1, \dots, l$; $\Omega_i \cap \Pi(m) \neq \emptyset$ and $\Omega_j \cap \Pi(m) = \emptyset$ implies $\sigma_m(i) < \sigma_m(j)$.

Theorem 5. *Consider an assignment problem with multiple donors and a partition of the set of available kidneys Π . A generalized age-based priority rule $\Psi : \mathcal{D}^\Pi \times \mathcal{K} \rightarrow \mathcal{A}^2$ satisfies individual rationality, 2-efficiency, and EGSP.*

Before concluding this section, it is worth pointing out that our rules are immune to other form of misrepresentation of the information about the set of donors a part from withholding some potential donors.

Remark 1. *Note that (generalized) age-based priority rules are also immune to manipulation by the introduction of dummy donors. That is, a patient does not improve by presenting a donor whose kidneys are not compatible with any other patient. That is, consider patient i and let $\Omega_i = \bar{\Omega}_i \cup \{\omega\}$. Let $\succ \in \mathcal{D}^\Pi$ be such that for each patient $j \neq i$, $\omega \notin D(\succ_j)$. Then, $\Psi(\succ, \Omega) = \Psi(\succ, (\Omega_{N \setminus \{i\}}, \bar{\Omega}_i))$.*

6 Conclusion

In this paper we provide a theoretical framework to design PKE protocols that encourage compatible pairs to participate into PKE program. The relatively low number of kidney paired exchanges performed in the US with respect to the number of transplants from living donors makes clear that there are still significant barriers to its utilization.

We believe that the participation of compatible pairs may represent the most important factor in expanding the number of kidney paired exchanges. As regards the US system, different proposals have been presented in order to increase the chance of an incompatible pair of finding a compatible match: the introduction of a nationwide PKE registry, the use of matching algorithms that include three-way matches and of non-directed (altruistic) donors. Some of these strategies can be carried out together, but others cannot. Our paper enlightens the difficulties in implementing incentive compatible algorithms that allow three-way matches in our age-based preference domain.

It is therefore crucial to predict the impact of allowing compatible pairs to participate to PKE programs, to understand which type of proposal may be more effective in increasing the number of transplants and their quality. A recent study by Gentry et al. (2007) uses simulated data to prove that there could be large benefits for both incompatible pairs and compatible pairs if compatible pairs were willing to participate to KPE programs. The reason why expanding the pool through participation by compatible pairs could be a very successful strategy, is mostly driven by the blood group imbalance in the pool of incompatible pairs. Most of group O-donors can directly donate to their intended recipients, and so group-O recipients in the KPE pool must rely on a scarce number of compatible donors and rarely can find a match. Participation of compatible pairs could dramatically reduce such imbalance and would nearly double the match rate for incompatible pairs (28.2 to 64.5 percent for a single-center program and 37.4 to 75.4 percent for a national program). Compatible pairs could also benefit from participation. Having defined the benefit of participation as the possibility of being matched with a donor at least ten years younger than the related donor (and not considering other potential

benefits as an increase in HLA matching),¹⁹ a single compatible pair would have a 34 percent chance of finding a better match in a single center program and 48 percent in a national registry. If all compatible pairs were willing to participate to a PKE program, then still 11.7 percent (single-center) and 14.7 percent of these compatible pairs would find a more favorable match.²⁰ In conclusion, Gentry et al. (2007) present an important supporting argument for the introduction of an algorithm, as we suggest in this paper, that encourages compatible pairs participation in PKE programs.

We would like to end the paper by commenting on further possible applications of our framework. Although our model is directly inspired by the structure of PKE problems, it applies to other problems of centralized allocation of indivisible goods without money. A pertinent example is the holidays vacant houses swap networks. These networks usually arrange pairwise swaps due to administrative restrictions. Agents' preferences may be based on the location or size and quality (facilities) of the house. Normally, different agents may consider different locations as attractive or not (so some house exchanges are not compatible). Every agent ranks the houses in each location, however, taking into account only the facilities of each house. So, while which location turns out be attractive for an agent is her private information, how agents rank houses in the same location it is easy to predict and such a ranking is the same for all agents interested in this location.

7 Appendix: Proofs

Proof of Theorem 1. The arguments follow the proof of Theorem 1 in Nicoló and Rodríguez-Álvarez (2009). We include the complete proof for the sake of completeness.

Let $3 \leq k \leq n - 1$. Assume, by way of contradiction, that there are a partition Π and a rule φ that satisfies *individual rationality*, *k-efficiency*, and *strategy-proofness* in

¹⁹Another potential benefit taken into consideration for female recipient is the reduction of high immunological risk donor/recipient combination (child-to-mother or husband-to-wife) due to exposure in uterus to paternal HLA antigens.

²⁰Since compatible pairs compete for a fixed number of incompatible pairs with young donors, the potential benefit of a compatible pairs of enrolling in a PKE program is decreasing in the number of compatible pairs participating in the program.

\mathcal{D}^Π . Without loss of generality, by our Assumption, let $\omega_1 \in \Pi(l)$, $\omega_2 \in \Pi(l')$, $\omega_k \in \Pi(\bar{l})$ and $\omega_{k+1} \in \Pi(\bar{l}')$ with $l < l'$ and $\bar{l} < \bar{l}'$.²¹ Let Table 1 represent patients' preferences over compatible kidneys according to the preference profile $\succsim \in \mathcal{D}^\Pi$.

\succsim_1	\succsim_2	\dots	\succsim_{k-1}	\succsim_k	\succsim_{k+1}
ω_2	ω_3	\dots	ω_k	ω_{k+1}	ω_1
ω_1	ω_2	\dots	ω_{k+1}	ω_k	ω_2
\dots	\dots	\dots	ω_{k-1}	\dots	ω_{k+1}
\dots	\dots	\dots	\dots	\dots	\dots

Table 1: \succsim , Theorem 1.

Let $\succsim' \in \mathcal{D}^\Pi$ be such that for each $i \neq k-1$, $\succsim_i = \succsim'_i$, and $D(\succsim'_{k-1}) = \{\omega_k\}$. Under profile \succsim' , by *individual rationality*, either no object is assigned to any patient $1, \dots, k+1$, or patient $k+1$ receives ω_2 , patient 1 receives ω_1 , and every other patient i receives ω_{i+1} (the kidney of her next to the right neighbor). By *k-efficiency*:

$$\varphi(\succsim') = \begin{bmatrix} (1, \omega_1), \\ (i, \omega_{i+1}), \quad \forall i = 2, \dots, k \\ (k+1, \omega_2) \end{bmatrix}.$$

By *strategy-proofness*, $\varphi_{k-1}(\succsim) \succsim_{k-1} \varphi_{k-1}(\succsim') = \omega_k$. Note that, according to \succsim_{k-1} , ω_k is patient $k-1$'s preferred kidney. Then, $\varphi_{k-1}(\succsim) = \omega_k$. By *k-efficiency* and *individual rationality*, $\varphi(\succsim) = \varphi(\succsim')$.

Let $\succsim'' \in \mathcal{D}^\Pi$ be such that for each $i \neq k+1$, $\succsim_j = \succsim''_j$ and $D(\succsim''_{k+1}) = \{\omega_1\}$. The same arguments we employed to determine $\varphi(\succsim')$ apply here to obtain:

$$\varphi(\succsim'') = \begin{bmatrix} (i, \omega_{i+1})(\text{modulo } k+1), \quad \forall i \notin \{k, k-1\} \\ (k-1, \omega_{k+1}), \\ (k, \omega_k) \end{bmatrix}.$$

²¹For instance, we can assume that $\omega_1 \in \Pi(1)$ and $\{\omega_2, \omega_k\} \in \Pi(2)$, and $\omega_{-k+1} \in \Pi(3)$. Alternatively, we can have $\{\omega_1, \omega_k\} \subseteq \Pi(1)$ and $\{\omega_2, \omega_{k+1}\} \subseteq \Pi(2)$, and apply a convenient relabeling of patients and donors in order to satisfy our notational assumption. (See page 7, definition of Π .)

Note that $\omega_1 = \varphi_{k+1}(\tilde{\lambda}'') = \varphi(\tilde{\lambda}''_{k+1}, \tilde{\lambda}_{-(k+1)}) \succ_{k+1} \varphi_{k+1}(\tilde{\lambda}) = \omega_2$, which contradicts *strategy-proofness*. \square

Proof of Proposition 1. Let $\tilde{\lambda} \in \mathcal{D}^{\Pi^*}$. Because preferences over desirable kidneys and each patients' donor kidney are strict, it is clear that the priority mechanism selects a unique assignment. Let $b \in \mathcal{A}^2$, $b \neq \psi^*(\tilde{\lambda})$. We prove that b is not in the core. Assume first that $b \notin \mathcal{I}(\tilde{\lambda})$. There is $j \in N$ such that $\omega_j \succ_j b_j$. Let $a \in \mathcal{A}^2$ such that $a_j = \omega_j$. Clearly, a weakly dominates b via coalition $\{j\}$. Next, assume that $b \in \mathcal{I}(\tilde{\lambda})$. Let i be the patient such that $\psi_i^*(\tilde{\lambda}) \neq b_i$ and for each $i' < i$, $\psi_{i'}^*(\tilde{\lambda}) = b_{i'}$. Note that $\psi^*(\tilde{\lambda}) \in \mathcal{M}_i^{\sigma^*}(\tilde{\lambda})$, for each $i' < i$, $\psi_{i'}^*(\tilde{\lambda}) = b_{i'}$, and $b \in \mathcal{I}(\tilde{\lambda})$ imply that $\psi_i^*(\tilde{\lambda}) \succ_i b_i$. Because preferences over acceptable kidneys are strict, $\psi_i^*(\tilde{\lambda}) \succ_i b_i$. Let $j \in N$ be such that $\psi_j^*(\tilde{\lambda}) = \omega_j$ (and $\psi_j^*(\tilde{\lambda}) = \omega_j$). Since for each $i' < i$, $\psi_{i'}^*(\tilde{\lambda}) = b_{i'}$, $\omega_i = \psi_j^*(\tilde{\lambda}) \succ_j b_j$. Hence, $\psi^*(\tilde{\lambda})$ weakly dominates b via $\{i, j\}$. Finally, we prove that there is no $b \in \mathcal{A}^2$ such that b weakly dominates $\psi^*(\tilde{\lambda})$. Assume to the contrary there is $b \in \mathcal{A}^2$ and $T \subseteq N$ such that b weakly dominates $\psi^*(\tilde{\lambda})$ via T . Define the set $T' \equiv \{i \in T, \text{ such that } b_i \succ_i \psi_i^*(\tilde{\lambda})\}$. We first prove that $1 \notin T'$. By the definition of the priority algorithm, for each $b \in \mathcal{A}^2$, such that $b_1 \succ_1 \psi_1^*(\tilde{\lambda})$, $b \notin \mathcal{I}(\tilde{\lambda})$. Hence, $1 \notin T'$. Analogously, for patient 2, for each $b \in \mathcal{A}^2$, such that $b_2 \succ_2 \psi_2^*(\tilde{\lambda})$, either $b \notin \mathcal{I}^2(\tilde{\lambda})$, or $b_2 = \psi_1^*(\tilde{\lambda})$. Let patient k be such that $\omega_k = b_2 = \psi_1^*(\tilde{\lambda})$ and therefore $\omega_1 \in D(\tilde{\lambda}_k)$ and $\psi_k^*(\tilde{\lambda}) = \omega_1$. Since $\omega_1 \succ_k \omega_2$, $k \notin T'$, and also $2 \notin T'$. Note that for each $i, j \in N$ and each $\tilde{\lambda}_i \in \mathcal{D}^{\Pi^*}$, if $\omega_j \succ_i \psi_i^*(\tilde{\lambda})$, then either $\omega_i \notin D(\omega_j)$, or $\psi_j^*(\tilde{\lambda}) = \omega_k$ for some $k < i$. Repeating the argument iteratively for each $i = 3, \dots, n$ we obtain that $T' = \{\emptyset\}$, which contradicts that b weakly dominates $\psi^*(\tilde{\lambda})$ via T . \square

Proof of Theorem 2. By definition ψ^* satisfies *individual rationality*, and *2-efficiency*. Hence we check that ψ^* satisfies *strategy-proofness* in \mathcal{D}^{Π^*} . Let $\tilde{\lambda} \in \mathcal{D}^{\Pi^*}$ and let $\tilde{\lambda}'_1 \in \mathcal{D}_1^{\Pi^*}$ be such that $\psi_1^*(\tilde{\lambda}'_1, \tilde{\lambda}_{-1}) \neq \psi_1^*(\tilde{\lambda})$. Without loss of generality, let $\psi_1^*(\tilde{\lambda}'_1, \tilde{\lambda}_{-1}) = \omega_j$ and $\psi_1^*(\tilde{\lambda}) = \omega_i$. If $i = 1$, then $D(\tilde{\lambda}_1) = \{\emptyset\}$ and $\omega_1 = \psi_1^*(\tilde{\lambda}) \succ_1 \psi_1^*(\tilde{\lambda}'_1, \tilde{\lambda}_{-1})$. If $j = 1$, by *individual rationality*, $\psi_1^*(\tilde{\lambda}) \in D(\tilde{\lambda}_1)$, and then $\psi_1^*(\tilde{\lambda}) \succ_1 \psi_1^*(\tilde{\lambda}'_1, \tilde{\lambda}_{-1})$. If $i > j$ and $i \neq 1, j \neq 1$, then by the definition of ψ^* , $\omega_j \notin (D(\tilde{\lambda}_1) \cup \{\omega_1\})$. Then, by *individual rationality*, $\psi_1^*(\tilde{\lambda}) \succ_1 \psi_1^*(\tilde{\lambda}'_1, \tilde{\lambda}_{-1})$. If $i < j$ and $i \neq 1, j \neq 1$, since

$\succsim_1 \in \mathcal{D}^{\Pi^*}$ and ψ^* satisfies *individual rationality*, $\psi_1^*(\succsim) \succ_1 \psi_1^*(\succsim'_1, \succsim_{-1})$. Hence, for each $\succsim'_1 \in \mathcal{D}^{\Pi^*}$, $\psi_1^*(\succsim) \succsim_1 \psi_1^*(\succsim'_1, \succsim_{-1})$. Let $i \in N$ be such that $\psi_1^*(\succsim) = \omega_i$ and $\psi_i^*(\succsim) = \omega_1$. Because ψ^* satisfies *individual rationality*, $\omega_1 \in D(\succsim_i)$. Since $\succsim_i \in \mathcal{D}_i^{\Pi^*}$, for each $\omega \in \Omega \setminus \{\omega_1\}$, $\psi_i^*(\succsim) \succ_i \omega$. Therefore, for each $\succsim'_i \in \mathcal{D}_i^{\Pi^*}$, $\psi_i^*(\succsim) \succsim_i \psi_i^*(\succsim'_i, \succsim_{-i})$. Consider now patient 2. By the definition of ψ^* , for each $i \in N$ such that $\omega_i \succ_2 \psi_2^*(\succsim)$, either $\omega_2 \notin D(\succsim_i)$ or $\omega_i = \psi_1^*(\succsim)$. Therefore, by the definition of ψ^* , for each $\succsim'_2 \in \mathcal{D}^{\Pi^*}$, $\psi_2^*(\succsim) \succsim_2 \psi_2^*(\succsim'_2, \succsim_{-2})$. Repeating the same argument as many times as necessary, we obtain that ψ^* satisfies *strategy-proofness* in \mathcal{D}^{Π^*} .

Next, we prove the necessity side. Let φ be a rule that satisfies *strategy-proofness*, *individual rationality*, and *2-efficiency*. Note that, for each $i \in N$, each $a \in \mathcal{A}^2$, and each $\succsim_i \in \mathcal{D}^{\Pi^*}$, $a_i \sim_i \omega_i$ if and only if $a_i = \omega_i$. Moreover, for each $i \in N$, each $a \in \mathcal{A}^2$, and each $\succsim_i \in \mathcal{D}^{\Pi^*}$ such that $a_i \succsim_i \omega_i$ there is $\succsim'_i \in \mathcal{D}^{\Pi^*}$ such that

- (i) for each $b \in \mathcal{A}^2 \setminus \{a\}$, $b_i \succsim_i a_i$ if and only if $b_i \succsim'_i a_i$,
- (ii) for each $b \in \mathcal{A}^2 \setminus \{a\}$, $a_i \succsim_i b_i$ if and only if $a_i \succsim'_i b_i$,
- (iii) for each $b \in \mathcal{A}^2 \setminus \{a\}$, $a_i \succ_i b_i$ if and only if $a_i \succ'_i b_i$, and $a_i \succ'_i \omega_i \succ'_i b_i$.

Hence, the domain \mathcal{D}^{Π^*} satisfies Assumptions A–B on the domain of preferences proposed by Sönmez (1999). By (Sönmez, 1999, Theorem 1), if there is a rule φ that satisfies *individual rationality*, *2-efficiency*, and *strategy-proofness* in \mathcal{D}^{Π^*} , for each $\succsim \in \mathcal{D}^{\Pi^*}$, φ selects an assignment in the (strict) core of the pairwise exchange problem associated to \succsim . We have just seen that ψ^* satisfies *individual rationality*, *2-efficiency*, and *strategy-proofness* in \mathcal{D}^{Π^*} . By Proposition 1, for each $\succsim \in \mathcal{D}^{\Pi^*}$, $\psi^*(\succsim)$ is the unique assignment in the core of the pairwise kidney exchange problem associated to \succsim . Therefore, $\varphi = \psi^*$. \square

The following remark and lemma are useful in the proof of Theorem 3. If the last element of the partition of the set of kidneys contains a unique kidney, then there are priority orderings that do not respect the partition but do define priority rules that are equivalent to an age-based priority rule. This may happen, when the position in the priority of patient with the worst kidney is switched with the patient that occupies the next to the last position in the priority ordering.

Remark 2. Let $\Pi = \{\Pi(1), \dots, \Pi(m)\}$ be a partition of Ω . If the priority ordering σ does not respect Π , then

- (i) either there are $i, j, k \in N$ $l, l' \leq m$ such that $\omega_i \in \Pi(l)$, $\omega_j \in \pi(l')$, $\sigma(i) > \sigma(j)$ and $\sigma(i) < \sigma(k)$,
- (ii) or there are $i', j' \in N$ such that $\Pi(m) = \{\omega_{i'}\}$, $\omega_{j'} \in \Pi(m-1)$, $\sigma(i') = (n-1)$ and $\sigma(j') = n$.

Lemma 2. Let $\Pi = \{\Pi(1), \dots, \Pi(m)\}$ be such that $\#\Pi(m) = 1$. Let the priority ordering σ respects Π and $i, j \in N$ such that $\{\omega_i\} = \Pi(m)$ and $\sigma(j) = (n-1)$. (Therefore, $\omega_j \in \Pi(m-1)$.) If the priority ordering σ' is such that for each $k \notin \{i, j\}$, $\sigma'(k) = \sigma(k)$, $\sigma'(i) = (n-1)$, and $\sigma'(j) = n$, then for each $\succsim \in \mathcal{D}^\Pi$, $\mathcal{M}_n^\sigma(\succsim) = \mathcal{M}_n^{\sigma'}(\succsim)$.

Proof. Note that for each $\succsim \in \mathcal{D}^\Pi$, and each $k < (n-1)$, $\mathcal{M}_k^\sigma(\succsim) = \mathcal{M}_k^{\sigma'}(\succsim)$. For each $k < (n-1)$ and each $a, a' \in \mathcal{M}_k^\sigma(\succsim)$, $a_k \sim_k a'_k$. Assume that $\psi_i^{\sigma'}(\succsim) \neq \{\omega_j\}$. Note that for each $k \notin \{i, j\}$ if $\{\omega_i, \omega_j\} \subseteq D(\succsim_k)$, then $\omega_j \succ_k \omega_i$. This implies that for each $a \in \mathcal{M}_{(n-1)}^{\sigma'}(\succsim)$ such that $a_j \neq \omega_j$ there is $a' \in \mathcal{M}_n^{\sigma'}(\succsim)$ with $a'_j = a_j$. (Patients i and j do not compete for other patients' donors, since other patients always prefer i 's donor.) Hence, $\mathcal{M}_n^\sigma(\succsim) = \mathcal{M}_n^{\sigma'}(\succsim)$. Finally, assume that $\psi_i^{\sigma'}(\succsim) = \{\omega_j\}$. Since $\psi^{\sigma'}(\succsim) \in \mathcal{I}(\succsim)$, $\omega_i \in D(\succsim_j)$. Assume to the contrary that $\psi_i^\sigma(\succsim) \neq \psi_i^{\sigma'}(\succsim)$. Since $\psi^{\sigma'}(\succsim) \in \mathcal{M}_{(n-2)}^\sigma(\succsim)$ and $\{\omega_i\} = \Pi(m)$, then $\psi_j^\sigma(\succsim) \neq \omega_j$. Define the sets of patients S_1, S_2 as the maximal sets (with respect to inclusion) such that $S_1 \cap S_2 = \emptyset$, $j \in S_1$, $\#S_1 = \#S_2$, for each $k_1 \in S_1$, $\omega_{k_1} \in \Pi(m-1)$, for each $k_2 \in S_2$ there is $k_1 \in S_1$ such that $\psi_{k_2}^\sigma(\succsim) = \omega_{k_1}$ and $\omega_{k_2} \in \Pi(l)$ for some $l \leq (m-1)$. Define analogously S'_1, S'_2 as the maximal sets (with respect to inclusion) such that $S'_1 \cap S'_2 = \emptyset$, $\#S'_1 = \#S'_2$, for each $k_1 \in S'_1$, $\omega_{k_1} \in \Pi(m-1)$, for each $k'_2 \in S'_2$ there is $k'_1 \in S'_1$ such that $\psi_{k'_2}^{\sigma'}(\succsim) = \omega_{k'_1}$ and $\omega_{k'_2} \in \Pi(l)$ for some $l \leq (m-1)$. Clearly $j \notin (S'_1 \cup S'_2)$. Because $\psi^\sigma(\succsim)$ and $\psi^{\sigma'}(\succsim)$ are pairwise assignments, either there is a patient k^* such that $k^* \in (S_1 \cup S_2) \setminus (S'_1 \cup S'_2)$, or there is a patient k^{**} such that $k^{**} \in (S'_1 \cup S'_2) \setminus (S_1 \cup S_2)$ (or both). Assume that there is a patient k^* such that $k^* \in (S_1 \cup S_2) \setminus (S'_1 \cup S'_2)$. Then, either $\psi_{k^*}^\sigma(\succsim) \succ_{k^*} \psi_{k^*}^{\sigma'}(\succsim)$, or $\psi_{k^*}^{\sigma'}(\succsim) \succ_{k^*} \psi_{k^*}^\sigma(\succsim)$. This implication however, contradicts the fact that for each $k < (n-1)$ and each $a, a' \in \mathcal{M}_k^\sigma(\succsim)$, $a_k \sim_k a'_k$. The remaining case generates an identical contradiction. Hence, if $\psi_i^{\sigma'}(\succsim) = \{\omega_j\}$, then $\psi_i^\sigma(\succsim) = \{\omega_j\}$ and $\mathcal{M}_n^\sigma(\succsim) = \mathcal{M}_n^{\sigma'}(\succsim)$. \square

Proof of Theorem 3. Consider a partition Π and a priority rule φ^σ that is not an age-based priority rule. Hence, the priority ordering σ does not respect Π and, by Remarks 2 and Lemma 2, there are $i, j, m \in N, k, k' \in \mathbb{N}$ be such that $\omega_i \in \Pi(k), \omega_j \in \Pi(k'), k \leq k', \sigma(i) > \sigma(j)$, and $\sigma_m > \sigma_j$. Let $\succsim \in \mathcal{D}^\Pi$ be such that $D(\succsim_i) = \{\omega_m\}, D(\succsim_j) = \{\omega_m\}$, and $D(\succsim_m) = \{\omega_i, \omega_j\}$. Clearly, $\psi_m^\sigma(\succsim) = \omega_j$. Let $\succsim'_m \in \mathcal{D}_m^\Pi$ be such that $D(\succsim'_m) = \{\omega_i\}$. Then, by 2-efficiency, $\psi_m^\sigma(\succsim'_m, \succsim_{-m}) = \omega_i$, and $\psi_m^\sigma(\succsim'_m, \succsim_{-m}) \succ_m \psi_m^\sigma(\succsim)$, which proves that ψ^σ violates *strategy-proofness*.

Next, we prove that if σ respects Π , then ψ^σ satisfies *group strategy-proofness*. In order to simplify notation, we consider the natural priority ordering σ^* . The arguments apply directly for every arbitrary priority ordering that respects Π . Assume to the contrary that ψ^* violates *group strategy-proofness*. Then, there is $T \subseteq N, \succsim \in \mathcal{D}^\Pi, \succsim'_T \in \times_{i \in T} \mathcal{D}_i^\Pi$ such that for each $i \in T, \psi_i^*(\succsim'_T, \succsim_{N \setminus T}) \succ_i \psi_i^*(\succsim)$. Let patient $j \in T$ be such that for each $k \in T, j \leq k$. Since $j \in T, \succsim_j \neq \succsim'_j$ and $\psi_j^*(\succsim'_T, \succsim_{N \setminus T}) \succ_j \psi_j^*(\succsim)$, necessarily $\psi_j^*(\succsim'_T, \succsim_{N \setminus T}) \neq \psi_j^*(\succsim)$. There are two possibilities:

- (i) For each $i < j, \psi_i^*(\succsim'_T, \succsim_{N \setminus T}) = \psi_i^*(\succsim)$ and for some patient $k, \omega_k = \psi_j^*(\succsim'_T, \succsim_{N \setminus T})$. By the definition of $\psi^*, \omega_j \in D(\succsim'_k) \setminus D(\succsim_k)$ and $\omega_k \succ_k \omega_j$. Because $\succsim_k \neq \succsim'_k, k \in T$. However, by ψ^* 's *individual rationality*, $\psi_k^*(\succsim) \succsim_k \omega_k \succ_k \omega_j = \psi_k^*(\succsim'_T, \succsim_{N \setminus T})$, which contradicts $k \in T$.
- (ii) There is $i < j$ such that $\psi_i^*(\succsim'_T, \succsim_{N \setminus T}) \neq \psi_i^*(\succsim)$. Let $i' < j$ be such that for each $i < i' \psi_i^*(\succsim'_T, \succsim_{N \setminus T}) = \psi_i^*(\succsim)$. If $\psi_{i'}^*(\succsim'_T, \succsim_{N \setminus T}) \succ_{i'} \psi_{i'}^*(\succsim)$, by the definition of age-based priority rule, there is $k \in T$ such that $\omega_{i'} \notin D(\succsim_k)$ and $\omega_{i'} \in D(\succsim'_k)$ and $\psi_{i'}^*(\succsim'_T, \succsim_{N \setminus T}) = \omega_k$. However, by *individual rationality*, $\psi_k^*(\succsim) \succsim_k \omega_k \succ_k \omega_{i'} = \psi_k^*(\succsim'_T, \succsim_{N \setminus T})$, which contradicts $k \in T$. Finally, if $\psi_{i'}^*(\succsim) \succ_{i'} \psi_{i'}^*(\succsim'_T, \succsim_{N \setminus T})$, then there is $k' \in T$ such that $\psi_{i'}^*(\succsim) = \omega_{k'}$. By the definitions of ψ^* and $i', \omega_{i'} \in D(\succsim_{k'}) \setminus D(\succsim'_{k'})$, and for each $\succsim_{k'} \in \mathcal{D}^\Pi, \omega_{i'} = \psi_{k'}^*(\succsim) \succ_{k'} \psi_{k'}^*(\succsim'_T, \succsim_{N \setminus T})$, which contradicts $k \in T$.

Because both cases exhaust all the possibilities, this suffices to prove *group strategy-proofness*. \square

Proof of Lemma 1. Assume to the contrary that there are $i, j \in N$ and $\succsim \in \mathcal{D}^\Pi$ such that $\omega_i \in D(\succsim_j)$ and $\omega_j \in D(\succsim_i)$ but $\omega_i \succ_j \varphi_j(\succsim)$ and $\omega_j \succ_i \varphi_i(\succsim)$. Let $\succsim' \in$

\mathcal{D}^Π be such that $\bar{\lambda}'_i = \bar{\lambda}_i$, $\bar{\lambda}'_j = \bar{\lambda}_j$, and for each $k \in N \setminus \{i, j\}$, $D(\bar{\lambda}'_k) = \{\varphi_k(\bar{\lambda})\}$. Let $k' \in N \setminus \{i, j\}$. By *individual rationality*, $\varphi_{k'}(\bar{\lambda}'_{k'}, \bar{\lambda}_{-k'}) \in \{\omega_{k'}, \varphi_{k'}(\bar{\lambda})\}$. By *strategy-proofness*, $\varphi_{k'}(\bar{\lambda}'_{k'}, \bar{\lambda}_{-k'}) \bar{\lambda}'_{k'} \varphi_{k'}(\bar{\lambda})$. Hence, $\varphi_{k'}(\bar{\lambda}'_{k'}, \bar{\lambda}_{-k'}) = \varphi_{k'}(\bar{\lambda})$, and by *non-bossiness*, $\varphi(\bar{\lambda}'_{k'}, \bar{\lambda}_{-k'}) = \varphi(\bar{\lambda})$. Repeating the argument with the remaining patients (one at a time), we obtain $\varphi(\bar{\lambda}') = \varphi(\bar{\lambda})$. Let $\bar{\lambda}'' \in \mathcal{D}^\Pi$ be such that $D(\bar{\lambda}''_i) = \{\omega_j, \varphi_i(\bar{\lambda})\}$, $D(\bar{\lambda}''_j) = \{\omega_i, \varphi_j(\bar{\lambda})\}$ and for each patient $k \notin \{i, j\}$, $\bar{\lambda}''_k = \bar{\lambda}'_k$. By *individual rationality* and *strategy-proofness*, $\varphi_i(\bar{\lambda}''_i, \bar{\lambda}'_{-i}) = \varphi_i(\bar{\lambda}')$. By *non-bossiness*, $\varphi(\bar{\lambda}''_i, \bar{\lambda}'_{-i}) = \varphi(\bar{\lambda}')$. Similarly, by *individual rationality* and *strategy-proofness*, $\varphi_j(\bar{\lambda}''_{\{i,j\}}, \bar{\lambda}'_{-\{i,j\}}) = \varphi_j(\bar{\lambda}''_i, \bar{\lambda}'_{-i})$. By *non-bossiness*, $\varphi(\bar{\lambda}''_{\{i,j\}}, \bar{\lambda}'_{-\{i,j\}}) = \varphi(\bar{\lambda}''_i, \bar{\lambda}'_{-i})$. Thus, $\varphi(\bar{\lambda}'') = \varphi(\bar{\lambda}')$. Let $\bar{\lambda} \in \mathcal{D}^\Pi$ be such that $D(\bar{\lambda}_i) = \{\omega_j\}$, $D(\bar{\lambda}_j) = \{\omega_i\}$, and for each patient $k \notin \{i, j\}$, $\bar{\lambda}_k = \bar{\lambda}''_k$. By *individual rationality* and *strategy-proofness*, $\varphi_i(\bar{\lambda}_i, \bar{\lambda}''_{-i}) = \{\omega_i\}$. By *individual rationality*, $\varphi_j(\bar{\lambda}_i, \bar{\lambda}''_{-i}) \in \{\omega_j, \varphi_j(\bar{\lambda}'')\}$. By *2-efficiency*, $\varphi_j(\bar{\lambda}_i, \bar{\lambda}''_{-i}) = \varphi_j(\bar{\lambda}'')$. Finally, by *individual rationality* and *strategy-proofness*, $\varphi_j(\bar{\lambda}_{\{i,j\}}, \bar{\lambda}''_{-\{i,j\}}) = \{\omega_j\}$ and $\varphi_i(\bar{\lambda}_{\{i,j\}}, \bar{\lambda}'_{-\{i,j\}}) = \{\omega_i\}$, which violates *2-efficiency*. \square

At this point we change the order in which we present the proofs with respect to the main text. First, we prove a general version of Theorem 4 (Theorem 6). Then, the proofs of Proposition 2 and Theorem 4 immediately follow from Theorem 6. We introduce some notation that generalizes the definition of *sequential matching maximizing rules* to arbitrary partitions Π of the set of donors.

For each $a \in \mathcal{A}^2$, each $j, k \in \mathbb{N}$ with $j \leq k$ and $k \leq l$, define:

$$M_{j,k}(a) \equiv \left\{ i \in N \left| \begin{array}{l} a_i \neq \omega_i \ \& \ \text{either } \omega_i \in \Pi(j) \text{ and } a_i \in \Pi(k) \\ \text{or } \omega_i \in \Pi(k) \text{ and } a_i \in \Pi(j) \end{array} \right. \right\}.$$

That is, $M_{j,k}(a)$ contains the patients with a donor in $\Pi(j)$ who receive a kidney in $\Pi(k)$ and the patients with a donor in $\Pi(k)$ who receive a kidney in $\Pi(j)$. For each $a \in \mathcal{A}^2$, let $P_{1,1}(a) \equiv \emptyset$ and define recursively for each $j, k \in N$, such that $\{j, k\} \neq \{1, 1\}$, $j \leq k$ and $k \leq l$:

$$P_{j,k}(a) \equiv \begin{cases} P_{j,k-1}(a) \cup M_{j,k-1}(a) & \text{if } j < k, \\ P_{j-1,l}(a) \cup M_{j-1,l}(a) & \text{if } j = k. \end{cases}$$

Theorem 6. *If φ satisfies individual rationality, 2-efficiency, strategy-proofness, and non-bossiness, then for each $\succsim \in \mathcal{D}^{\Pi}$, each $j, k \in \mathbb{N}$ with $j \leq k$ and $k \leq l$,*

$$\#M_{j,k}(\varphi(\succsim)) \geq \#M_{j,k}(a),$$

for each $a \in \mathcal{I}(\succsim)$ such that for each $i \in P_{j,k}(\varphi(\succsim))$, $a_i = \varphi_i(\succsim)$.

Proof of Theorem 6. We prove first the result for $j = 1$ and $k = 1$. The arguments can be replicated iteratively to prove the general result.

Assume to the contrary that φ satisfies *individual rationality, 2-efficiency, strategy-proofness, and non-bossiness* but there is $\succsim \in \mathcal{D}^{\Pi}$ and an assignment $a \in \mathcal{I}(\succsim)$ such that $\#M_{1,1}(a) > \#M_{1,1}(\varphi(\succsim))$. (By definition, for each $a' \in \mathcal{A}^2$, $P_{1,1}(a') = \emptyset$.) Let $b = \varphi(\succsim)$. Without any loss of generality, there is a set $T \subset N$ and $m, m' \in N \setminus T$ such that for each $i \in T \cup \{m, m'\}$, $\omega_i \in \Pi(1)$, $\varphi_m(\succsim) \notin \Pi(1)$, $\varphi_{m'}(\succsim) \notin \Pi(1)$ and:

- (i) For each $h \in T$, there is $h' \in T \setminus \{h\}$ such that $\varphi_h(\succsim) = \omega_{h'}$.
- (ii) For each $g \in T \cup \{m, m'\}$, there is $g' \in (T \cup \{m, m'\}) \setminus \{g\}$ such that $a_g = \omega_{g'}$.

Put Figure 1 about here.

Assume first that $T = \emptyset$. In this case, $a_m = \omega_{m'}$ and $a_{m'} = \omega_m$. Since $a \in \mathcal{I}(\succsim)$, $\omega_m \in D(\succsim_{m'})$ and $\omega_{m'} \in D(\succsim_m)$. Because $\varphi_m(\succsim) \notin \Pi(1)$ and $\varphi_{m'}(\succsim) \notin \Pi(1)$, this contradicts Lemma 1.

Assume $T \neq \emptyset$. Let $\succsim' \in \mathcal{D}^{\Pi}$ be such that for each $i \notin T \cup \{m, m'\}$, $D(\succsim'_i) = \varphi_i(\succsim)$ and for each $i' \in T \cup \{m, m'\}$, $\succsim'_{i'} = \succsim_{i'}$. Let $i \notin T \cup \{m, m'\}$. By *individual rationality*, $\varphi_i(\succsim'_i, \succsim_{-i}) \in \{\omega_i, \varphi_i(\succsim)\}$. By *strategy-proofness*, $\varphi_i(\succsim'_i, \succsim_{-i}) \succsim'_i \varphi_i(\succsim)$. Then, $\varphi_i(\succsim'_i, \succsim_{-i}) = \varphi_i(\succsim)$ and by *non-bossiness*, $\varphi(\succsim'_i, \succsim_{-i}) = \varphi(\succsim)$. Repeating the same argument exchanging the preference of each patient, we obtain $\varphi(\succsim') = \varphi(\succsim)$. Let $\succsim''_m \in \mathcal{D}^{\Pi}_m$ be such that $D(\succsim''_m) = a_m \in \Pi(1)$. By *individual rationality*, $\varphi_m(\succsim''_m, \succsim'_{-m}) \in \{\omega_m, a_m\}$. By *strategy-proofness*, $\varphi_m(\succsim') \succsim'_m \varphi_m(\succsim''_m, \succsim'_{-m})$. Hence, $\varphi_m(\succsim''_m, \succsim'_{-m}) = \omega_m$. We have to consider three cases.

- (i) $\varphi_{m'}(\succsim''_m, \succsim'_m) = \omega_{m'}$. Let $b \in \mathcal{A}^2$ be such that for each $k \in T \cup \{m, m'\}$, $b_k = a_k$ and for each $i \notin T \cup \{m, m'\}$, $b_i = \varphi_i(\succsim''_m, \succsim'_m)$. Note the assignment b is such that all the

patients in $T \cup \{m, m'\}$ receive a kidney in $\Pi(1)$. Then, for each $i' \in N \setminus \{m\}$, $b_{i'} \succ'_{i'} \varphi_{i'}(\tilde{\lambda}''_m, \tilde{\lambda}'_{-m})$, $b_m \succ''_m \varphi_m(\tilde{\lambda}''_m, \tilde{\lambda}'_m)$, and $b_{m'} \succ'_{m'} \varphi_{m'}(\tilde{\lambda}''_m, \tilde{\lambda}'_m)$ which contradicts *2-efficiency*.

(ii) $\varphi_{m'}(\tilde{\lambda}''_m, \tilde{\lambda}'_{-m}) \in \Pi(1) \setminus \{\omega_{m'}\}$. By *individual rationality* and the definition of the profile $(\tilde{\lambda}''_m, \tilde{\lambda}'_{-m})$, there are $i \in T$, such that $\varphi_i(\tilde{\lambda}''_m, \tilde{\lambda}'_{-m}) = \omega_i$. Consider the assignment $b \in \mathcal{A}^2$ defined in the previous paragraph. For each $i' \in N \setminus \{m\}$, $b_{i'} \succ'_{i'} \varphi_{i'}(\tilde{\lambda}''_m, \tilde{\lambda}'_{-m})$, $b_m \succ''_m \varphi_m(\tilde{\lambda}''_m, \tilde{\lambda}'_m)$, and $b_i \succ'_i \varphi_i(\tilde{\lambda}''_m, \tilde{\lambda}'_m)$, which contradicts *2-efficiency*.

(iii) $\varphi_{m'}(\tilde{\lambda}''_m, \tilde{\lambda}'_m) = \varphi_n(\tilde{\lambda}')$. Let $\tilde{\lambda}''_{m'} \in \mathcal{D}^\Pi_{m'}$ be such that $D(\tilde{\lambda}''_{m'}) = \{a_{m'}\}$. By an already familiar argument, *individual rationality* and *strategy-proofness*, imply that $\varphi_{m'}(\tilde{\lambda}''_{\{m, m'\}}, \tilde{\lambda}'_{-\{m, m'\}}) = \omega_{m'}$. By *individual rationality* and the definition of the profile $(\tilde{\lambda}''_{\{m, m'\}}, \tilde{\lambda}'_{-\{m, m'\}})$, there are at least two patients $i, i' \in T \cup \{m, m'\}$, such that $\varphi_i(\tilde{\lambda}''_{\{m, m'\}}, \tilde{\lambda}'_{-\{m, m'\}}) = \omega_i$ and $\varphi_{i'}(\tilde{\lambda}''_{\{m, m'\}}, \tilde{\lambda}'_{-\{m, m'\}}) = \omega_{i'}$, which implies a contradiction with *2-efficiency*.

Next, and just for the sake of completeness, we replicate the arguments to prove the result for $j = 1$ and $k = 2$. Assume to the contrary that φ satisfies *individual rationality*, *2-efficiency*, *strategy-proofness*, and *non-bossiness* but there is $\tilde{\lambda} \in \mathcal{D}^\Pi$ and an assignment $a \in \mathcal{I}(\tilde{\lambda})$ such that for each $i \in M_{1,1}(\varphi(\tilde{\lambda})) = P_{1,1}(\varphi(\tilde{\lambda}))$, $\varphi_i(\tilde{\lambda}) = a_i$ but

$$\#M_{1,2}(a) > \#M_{1,2}(\varphi(\tilde{\lambda})).$$

Without loss of generality there is a set $T \subset N$ and a pair of patients $m, m' \in N \setminus T$ such that for each $i \in T \cup \{m, m'\}$, $\omega_i \in (\Pi(1) \cup \Pi(2))$, $\omega_m \in \Pi(1)$, $\omega_{m'} \in \Pi(2)$, $\varphi_m(\tilde{\lambda}) \notin (\Pi(1) \cup \Pi(2))$, $\varphi_{m'}(\tilde{\lambda}) \notin \Pi(1)$, and:

- (i) For each $h \in T$ with $\omega_h \in \Pi(1)$ there is $h' \in T$ with $\omega_{h'} \in \Pi(2)$ such that $\varphi_h(\tilde{\lambda}) = \omega_{h'}$.
- (ii) For each $g \in T \cup \{m, m'\}$ with $\omega_g \in \Pi(1)$ there is $g' \in T \cup \{m, m'\}$ with $\omega_{g'} \in \Pi(2)$ such that $a_g = \omega_{g'}$.

Assume first that $T = \emptyset$. Then, repeating the argument in the proof for $j = 1$ and $k = 1$, we find a contradiction with Lemma 1.

Assume $T \neq \emptyset$. Let $\succsim' \in \mathcal{D}^\Pi$ be such that for each $i \notin T \cup \{m, m'\}$, $D(\succsim'_i) = \varphi_i(\succsim)$. By repeated application of *individual rationality*, *strategy-proofness*, and *non-bossiness*, $\varphi(\succsim') = \varphi(\succsim)$. Let $\succsim''_m \in \mathcal{D}^\Pi_m$ be such that $D(\succsim''_m) = a_m \in \Pi(2)$. The arguments in the proof of Lemma 2 imply $\varphi_m(\succsim''_m, \succsim'_{-m}) = \omega_m$. We have to consider three cases that mimic the proof of the case $j = 1$ and $k = 1$.

- (i) $\varphi_{m'}(\succsim''_m, \succsim'_{-m}) = \omega_{m'}$. Let $b \in \mathcal{A}^2$ be such that for each $i \in T \cup \{m, m'\}$, $b_i = a_i$ and for each $i' \notin T \cup \{m, m'\}$, $b_{i'} = \varphi_{i'}(\succsim''_m, \succsim'_m)$. Note that, for each $i \in T \cup \{m, m'\}$ with $\omega_i \in \Pi(1)$, $b_i \in \Pi(2)$ and for each $i' \in T \cup \{m, m'\}$ with $\omega_{i'} \in \Pi(2)$, $b_{i'} \in \Pi(1)$. Hence, for each $i'' \in N \setminus \{m\}$, $b_{i''} \succsim'_{i''} \varphi_{i''}(\succsim''_m, \succsim'_{-m})$, $b_m \succ''_m \varphi_m(\succsim''_m, \succsim'_m)$, and $b_{m'} \succ'_{m'} \varphi_{m'}(\succsim''_m, \succsim'_m)$ which contradicts *2-efficiency*.
- (ii) $\varphi_{m'}(\succsim''_m, \succsim'_{-m}) \in \Pi(1)$. By *individual rationality* and the definition of the profile $(\succsim''_m, \succsim'_{-m})$, there is $i \in T$ with $\omega_i \in \Pi(2)$ such that $\varphi_i(\succsim''_m, \succsim'_{-m}) = \omega_i$. The arguments of the previous paragraph immediately lead to a contradiction with *2-efficiency*.
- (iii) $\varphi_{m'}(\succsim''_m, \succsim'_{-m}) = \varphi_{m'}(\succsim')$. Combining the arguments in the previous paragraphs with the arguments in the proof of the case $j = 1$ and $k = 1$, we obtain a contradiction with *2-efficiency*.

We can sequentially apply the same arguments to prove the result for $j = 1$ and $k = 3, \dots, l$. Then, the arguments in the proof for $j = 1$ and $k = 1$ directly apply to the case $j = 2$ and $k = 2$, and iteratively to all the remaining steps. \square

Proof of Theorem 4. Note that Theorem 4 is just a special case of Theorem 6 for partitions consisting of two elements $\Pi = \{\Pi(1), \Pi(2)\}$. \square

Proof of Proposition 2. Assume to the contrary that φ satisfies *individual rationality*, *2-efficiency*, *strategy-proofness*, and *non-bossiness*, and there are $a \in \mathcal{A}^2$, $T \subset N$ and $\succsim \in \mathcal{D}^\Pi$ such that a strictly dominates $\varphi(\succsim)$ via T . Note that for each $i \in M_{1,1}(\varphi(\succsim))$ and each $\omega \in \Omega$, since $\succsim_i \in \mathcal{D}^\Pi$, $\varphi_i(\succsim) \succsim_i \omega$. Then, $i \notin T$. Analogously, for each $k = 2, \dots, l$, if $j \in M_{1,k}(\varphi(\succsim))$ and $\omega_j \in \Pi(k)$, $j \notin T$.

Next, assume that there is $j \in M_{1,2}(\varphi(\succsim))$ with $\omega_j \in \Pi(1)$ and $j \in T$. Since $\varphi_j(\succsim) \in \Pi(2)$ and a blocks φ via T , there is $k \in T$ such that $a_j = \omega_k \in \Pi(1)$. Since

$k \in T$, $a_k = \omega_j \succ_k \varphi_k(\zeta)$. Finally, let $b \in \mathcal{I}(\zeta)$ be such that for each $i \in M_{1,1}(\varphi(\zeta))$, $b_i = \varphi_i(\zeta)$, $b_j = a_j$, $b_k = a_k$, and for each $m \notin M_{1,1}(\varphi(\zeta)) \cup \{j, k\}$, $a_m = \omega_m$. Clearly, $b \in \mathcal{I}(\zeta)$ and $\#M_{1,1}(b) > \#M_{1,1}(\varphi(\zeta))$, which contradicts Theorem 6. Similarly, assume there is $j \in M_{1,3}(\varphi(\zeta))$ with $\omega_j \in \Pi(1)$ and $j \in T$. Since $\varphi_j(\zeta) \in \Pi(3)$ and a blocks φ via T , there is $k \in T$ such that $a_j = \omega_k \in \cup(\Pi(1) \cup \Pi(2))$. By the previous argument $\omega_k \in \Pi(2)$. Since $k \in T$, $a_k = \omega_j \succ_k \varphi_k(\zeta)$. Finally, let $b \in \mathcal{I}(\zeta)$ be such that for each $i \in (M_{1,1}(\varphi(\zeta)) \cup M_{1,2}(\varphi(\zeta)))$, $b_i = \varphi_i(\zeta)$, $b_j = a_j$, $b_k = a_k$, and for each $m \notin (M_{1,1}(\varphi(\zeta)) \cup M_{1,2}(\varphi(\zeta))) \cup \{j, k\}$, $a_m = \omega_m$. Clearly, $b \in \mathcal{I}(\zeta)$, for each $j \in M_{1,1}(\varphi(\zeta))$, $b_i = \varphi_i(\zeta) =$ but

$$\# \left\{ i \in N \left| \begin{array}{l} \text{either } \omega_i \in \Pi(1), b_i \in \Pi(2), \\ \text{or } \omega_i \in \Pi(2), b_i \in \Pi(1) \end{array} \right. \right\} > \#M_{1,2}(\varphi(\zeta)),$$

which contradicts Theorem 6. We can iteratively repeat the same argument as many times as necessary to show that there is no $i \in T$ with $\omega_i \in \Pi(1)$. The reasoning also apply for the remaining elements of the partition Π , to show that $T = \emptyset$. \square

Proof of Theorem 5. Let Ψ be a *generalized age-based priority rule* with permutations $\Sigma = \{\sigma_1, \dots, \sigma_l\}$. *Individual rationality* for generalized rules directly follows from the definition of generalized age-based priority rule. For *2-efficiency*, note that at each step of the algorithm a patient chooses her best preferred assignments in a set. Moreover, the kidneys that are *a priori* more desirable are the first kidneys to be assigned. We thus focus on EGSP. For each $\zeta \in \mathcal{D}^\Pi$, and each $\mathbf{K}' \in \mathcal{K}$ such that for each patient i , $K'_i \subset \Omega_i$, because $\mathcal{I}(\zeta, \mathbf{K}') \subseteq \mathcal{I}(\zeta, \Omega)$, by the iterative definition of Ψ , for every patient i , $\Psi_i(\zeta, \Omega) \succeq_i \Psi(\zeta, \mathbf{K}')$. Then, for each $T \subseteq N$, each $\zeta \in \mathcal{D}^\Pi$, each $\mathbf{K}'_T \in \times_{i \in T} \mathcal{K}_i$, for each $i \in T$:

$$\Psi_i(\zeta, \Omega) \succeq_i \varphi_i(\zeta, (\Omega_{N \setminus T}, \mathbf{K}'_T)). \quad (1)$$

Repeating the arguments in the proof of Theorem 3, we obtain that there are no $T \subset N$, $\zeta \in \mathcal{D}^\Pi$, $\zeta'_T \in \mathcal{D}^\Pi_T$ and $\mathbf{K}' \in \mathcal{K}$, such that for each $i \in T$:

$$\Psi_i(\zeta'_T, \zeta_{N \setminus T}, \mathbf{K}') \succ_i \Psi_i(\zeta, \mathbf{K}'). \quad (2)$$

Combining equations (1) and (2), and letting $\mathbf{K}' = (\Omega_{N \setminus T}, \mathbf{K}'_T)$, we obtain that there are

no $T \subset N$, $\succsim \in \mathcal{D}^{\Pi}$, $\succsim'_T \in \mathcal{D}_T^{\Pi}$, and $\mathbf{K}'_T \in \times_{i \in T} \mathcal{K}_i$ such that for each $i \in T$

$$\Psi_i(\succsim'_T, \succsim_{N \setminus T}, (\boldsymbol{\Omega}_{N \setminus T}, \mathbf{K}'_T)) \succ_i \Psi_i(\succsim, \boldsymbol{\Omega}).$$

□

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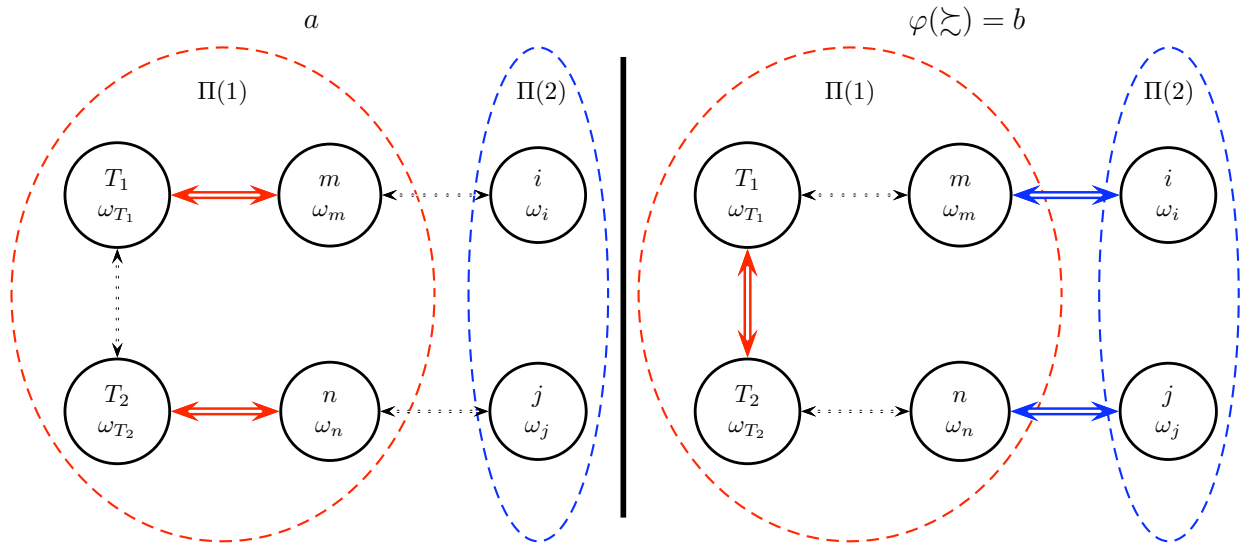


Figure 1: Proof of Theorem 6, $T = \{T_1, T_2\}$, $a \in \mathcal{I}(\zeta)$, $\varphi(\zeta) = b$.

8 Appendix B: NOT FOR PUBLICATION

In this section we briefly discuss the alternative framework in which the compatibility between donors and patients is public information. For each patient i let \mathbf{C}_i be i 's set of compatible kidneys. That is, the kidneys that according to the medical test carried by the transplant coordinator will not be rejected by i 's immune system and that may result in a successful transplantation. Let $\mathbf{C} \equiv \{\mathbf{C}_i\}_{i \in N}$. If the patients are allowed to reject any compatible kidney, we could define the sets of desirable (and non-desirable kidneys) just as those compatible kidneys that are preferred to the own donor kidney. With such definition, all our previous results would immediately follow.²² When the compatibility information is publicly observable, however, the arbitrary rejection of compatible kidneys does not seem adequate in the age-based environment. Following Nicoló and Rodríguez-Álvarez (2009), it seems reasonable to assume that whenever a patient considers a compatible kidney as desirable, then she also considers all the compatible kidneys from younger donors as desirable. In this framework, the only relevant information that remains private is the minimal quality (maximal age) that a patient requires to undergo transplantation. This observation leads to an additional restriction of patients' preferences.

Let denote by $\Pi = \{\Pi(1), \dots, \Pi(l)\}$ be a partition of Ω .

The preference relation $\succsim_i \in \mathcal{P}$ is a Π -based *preference consistent with \mathbf{C}* if

- (i) for each $\omega \in D(\succsim_i)$ with $\omega \in \Pi(m)$, and for each $\omega' \in \mathbf{C}_i \setminus \{\omega_i\}$ such that $\omega' \in \Pi(m')$ and $m' \leq m$, $\omega' \in D(\succsim_i)$.
- (ii) $\omega_i \in \mathbf{C}_i$ and $\omega_i \in \Pi(j)$ imply that for each $\omega' \in \Pi(j')$ with $j < j'$, $\omega' \in ND(\succsim_i)$.
- (iii) for each $\omega, \omega' \in D(\succsim_i)$, $\omega \in \Pi(j)$ and $\omega' \in \Pi(k)$, $\omega \succsim_i \omega'$ if and only if $j \leq k$.
- (iv) for each $\bar{\omega}, \bar{\omega}' \in ND(\succsim_i)$, $\omega_i \succ_i \bar{\omega}$ and $\bar{\omega} \sim_i \bar{\omega}'$.

We denote by $\mathcal{D}_i^\Pi(\mathbf{C})$ the domain of Π -based preferences consistent with \mathbf{C} for patient i and $\mathcal{D}^\Pi(\mathbf{C}) \equiv \times_i \mathcal{D}_i^\Pi(\mathbf{C})$.

²²Arbitrary rejections may be originated by any kind of preconception, or they may incorporate at some extent the preferences of the donor. For instance, the donor may accept the non-related living donation, if no long travels are required.

If \mathbf{C} is public information, the ranking of desirable kidneys is determined by Π , and each patient's set of desirable kidneys depends both on π and \mathbf{C} . By (i), if a compatible kidney is desirable for patient i , then all the compatible kidneys from a donor with lower (or equal) age is also desirable. By (ii), if patient i 's donor is compatible with patient i , then the kidneys from compatible donors older than patient i 's donor are not desirable. Items (iii) and (iv) just reproduce the notions of age-based preferences presented in section 3.

Note that if the information about compatibility is public, the only information about each patient's preference that remains private is the older compatible kidney that she is willing to receive. Note that our definition incorporates the possibility of altruistic motivations. A patient may have a compatible donor, but she may prefer to receive compatible kidneys that belong to the same element of the partition Π . (See Sönmez and Ünver (2010).) Alternatively, if she prefers her donor's kidney to compatible kidneys in the same element of the partition, the patient accepts an exchange only if she improves upon her donor's kidney. For each patient i , each partition Π , and each \mathbf{C} , $\mathcal{D}_i^\Pi(\mathbf{C}) \subset \mathcal{D}_i^\Pi$. On the other hand, the transplant coordinator may use the information in \mathbf{C} in the definition of the rule. Hence, we deal with \mathbf{C} specific rules, $\varphi^\mathbf{C} : \mathcal{D}^\Pi(\mathbf{C}) \rightarrow \mathcal{A}^k$. With the introduction of new notation, we can now state new versions of our results for specific \mathbf{C} .

Theorem 1'. *For each partition Π and each $k \in \mathbb{N}$ such that $3 \leq k \leq n - 1$, there are \mathbf{C} no rule $\varphi^\mathbf{C} : \mathcal{D}^\Pi(\mathbf{C}) \rightarrow \mathcal{A}^k$ satisfies individual rationality, k -efficiency, and strategy-proofness.*

Proof. The arguments in the proof of Theorem 1 apply directly. For each patient i , let $\mathbf{C}_i = \{\omega_{i+1}, \omega_{i+2}\}$ (modulo n). Note that all the preferences we used in the proof of Theorem 1 belong to $\mathcal{D}^\Pi(\mathbf{C})$. \square

Since for each \mathbf{C} , $\mathcal{D}_i^\Pi(\mathbf{C}) \subset \mathcal{D}_i^\Pi$, Proposition 1 applies and under the partition Π^* , the *age-based priority rule* always selects the unique core-stable assignment.

Theorem 2'. *For each \mathbf{C} , a rule $\varphi^\mathbf{C} : \mathcal{D}^{\Pi^*}(\mathbf{C}) \rightarrow \mathcal{A}^2$ satisfies individual rationality, 2-efficiency and strategy-proofness if and only if $\varphi^\mathbf{C}$ is the age-based priority rule ψ^* .*

Proof. The arguments in the proof of Theorem 2 apply directly once we observe that the domain $\mathcal{D}^{\Pi^*}(\mathbf{C})$ satisfies Sönmez (1999)'s conditions. \square

With Theorem 2' at hand and with the fact that $\mathcal{D}_i^{\Pi}(\mathbf{C}) \subset \mathcal{D}_i^{\Pi}$, we obtain Theorem 3'.

Theorem 3'. *For each partition Π , each priority ordering σ , the priority rule ψ^σ satisfies strategy-proofness in $\mathcal{D}^{\Pi}(\mathbf{C})$ for each \mathbf{C} if and only if ψ^σ is an age-based priority rule. Moreover, if σ respects Π , then the age-based priority rule ψ^σ satisfies group strategy-proofness in $\mathcal{D}^{\Pi}(\mathbf{C})$.*

Proof. Consider a partition Π and a priority ordering σ that does not respect Π . Hence, the priority ordering σ does not respect Π and, by Remark 2, there are $i, j, m \in N$, $k, k' \in \mathbb{N}$ be such that $\omega_i \in \Pi(k)$, $\omega_j \in \Pi(k')$, $k \leq k'$, $\sigma(i) > \sigma(j)$, and $\sigma_m > \sigma_j$. Let \mathbf{C} and $\succsim \in \mathcal{D}^{\Pi}$ be such that $\mathbf{C}_i = D(\succsim_i) = \{\omega_m\}$, $\mathbf{C}_j = D(\succsim_j) = \{\omega_m\}$, and $\mathbf{C}_m D(\succsim_m) = \{\omega_i, \omega_j\}$. Clearly, $\psi_m^\sigma(\succsim) = \omega_j$. Let $\succsim'_m \in \mathcal{D}_m^{\Pi}$ be such that $D(\succsim'_m) = \{\omega_i\}$. Then, $\psi_m^\sigma(\succsim'_m, \succsim_{-m}) = \omega_i$, and $\psi_m^\sigma(\succsim'_m, \succsim_{-m}) \succsim_m \psi_m^\sigma(\succsim)$, which proves that ψ^σ violates strategy-proofness. Since $\mathcal{D}^{\Pi}(\mathbf{C}) \subset \mathcal{D}^{\Pi}$, the arguments in the proof of Theorem 3 prove that every age-based priority rule satisfies group strategy-proofness in $\mathcal{D}^{\Pi}(\mathbf{C})$. \square

We conclude with the generalization of Lemma 1 and example to the new environment.

Lemma 1'. *There are \mathbf{C} such that for each $\varphi^{\mathbf{C}} : \mathcal{D}^{\Pi}(\mathbf{C}) \rightarrow \mathcal{A}^2$ that satisfies individual rationality, 2-efficiency, strategy-proofness, and non-bossiness then for each $i, j \in N$ and $\succsim \in \mathcal{D}^{\Pi}(\mathbf{C})$, $\omega_i \in D(\succsim_j)$ and $\omega_j \in D(\succsim_i)$ imply either $\varphi_i^{\mathbf{C}}(\succsim) \succsim_i \omega_j$ or $\varphi_j^{\mathbf{C}}(\succsim) \succsim_j \omega_i$ (or both).*

Proof. Assume there are $\tilde{\mathbf{C}}$, a partition Π , and $\varphi^{\tilde{\mathbf{C}}} : \mathcal{D}^{\Pi}(\tilde{\mathbf{C}})$ such that that there are $i, j \in N$ and $\tilde{\succsim} \in \mathcal{D}^{\Pi}(\tilde{\mathbf{C}})$ satisfying that $\omega_i \in D(\tilde{\succsim}_j)$ and $\omega_j \in D(\tilde{\succsim}_i)$ but $\omega_i \not\succsim_j \varphi_j^{\tilde{\mathbf{C}}}(\tilde{\succsim})$ and $\omega_j \not\succsim_i \varphi_i^{\tilde{\mathbf{C}}}(\tilde{\succsim})$. Define \mathbf{C} in such a way that $\mathbf{C}_i = \{\omega_j, \varphi_i^{\tilde{\mathbf{C}}}(\tilde{\succsim})\}$, $\mathbf{C}_j = \{\omega_i, \varphi_j^{\tilde{\mathbf{C}}}(\tilde{\succsim})\}$, and for each $k \in N \setminus \{i, j\}$ $\mathbf{C}_k = \varphi_k^{\tilde{\mathbf{C}}}(\tilde{\succsim})$. From now on, the proof literally replicates the arguments in the proof of Lemma 1. Assume to the contrary that there are $\varphi^{\mathbf{C}} : \mathcal{D}^{\Pi}(\mathbf{C}) \rightarrow \mathcal{A}^2$ and $\succsim \in \mathcal{D}^{\Pi}(\mathbf{C})$ such that $\omega_i \in D(\succsim_j)$ and $\omega_j \in D(\succsim_i)$ but $\omega_i \not\succsim_j \varphi_j^{\mathbf{C}}(\succsim)$ and $\omega_j \not\succsim_i \varphi_i^{\mathbf{C}}(\succsim)$. Let $\succsim' \in \mathcal{D}^{\Pi}(\mathbf{C})$ be such that $\succsim'_i = \succsim_i$, $\succsim'_j = \succsim_j$, and

for each $k \in N \setminus \{i, j\}$, $D(\tilde{\lambda}'_k) = \{\varphi_k^{\mathbf{C}}(\tilde{\lambda})\}$. Let $k' \in N \setminus \{i, j\}$. By *individual rationality*, $\varphi_{k'}^{\mathbf{C}}(\tilde{\lambda}'_{k'}, \tilde{\lambda}_{-k'}) \in \{\omega_{k'}, \varphi_{k'}^{\mathbf{C}}(\tilde{\lambda})\}$. By *strategy-proofness*, $\varphi_{k'}^{\mathbf{C}}(\tilde{\lambda}'_{k'}, \tilde{\lambda}_{-k'}) \tilde{\lambda}'_{k'} \varphi_{k'}^{\mathbf{C}}(\tilde{\lambda})$. Hence, $\varphi_{k'}^{\mathbf{C}}(\tilde{\lambda}'_{k'}, \tilde{\lambda}_{-k'}) = \varphi_{k'}^{\mathbf{C}}(\tilde{\lambda})$, and by *non-bossiness*, $\varphi^{\mathbf{C}}(\tilde{\lambda}'_{k'}, \tilde{\lambda}_{-k'}) = \varphi^{\mathbf{C}}(\tilde{\lambda})$. Repeating the argument with the remaining patients (one at a time), we obtain $\varphi^{\mathbf{C}}(\tilde{\lambda}') = \varphi^{\mathbf{C}}(\tilde{\lambda})$. Let $\tilde{\lambda}'' \in \mathcal{D}^{\Pi}(\mathbf{C})$ be such that $D(\tilde{\lambda}''_i) = \{\omega_j, \varphi_i^{\mathbf{C}}(\tilde{\lambda})\}$, $D(\tilde{\lambda}''_j) = \{\omega_i, \varphi_j^{\mathbf{C}}(\tilde{\lambda})\}$ and for each patient $k \notin \{i, j\}$, $\tilde{\lambda}''_k = \tilde{\lambda}'_k$. By *individual rationality* and *strategy-proofness*, $\varphi_i^{\mathbf{C}}(\tilde{\lambda}''_i, \tilde{\lambda}'_{-i}) = \varphi_i^{\mathbf{C}}(\tilde{\lambda}')$. By *non-bossiness*, $\varphi^{\mathbf{C}}(\tilde{\lambda}''_i, \tilde{\lambda}'_{-i}) = \varphi^{\mathbf{C}}(\tilde{\lambda}')$. Similarly, by *individual rationality* and *strategy-proofness*, $\varphi_j^{\mathbf{C}}(\tilde{\lambda}''_{\{i,j\}}, \tilde{\lambda}'_{-\{i,j\}}) = \varphi_j^{\mathbf{C}}(\tilde{\lambda}''_i, \tilde{\lambda}'_{-i})$. By *non-bossiness*, $\varphi^{\mathbf{C}}(\tilde{\lambda}''_{\{i,j\}}, \tilde{\lambda}'_{-\{i,j\}}) = \varphi^{\mathbf{C}}(\tilde{\lambda}''_i, \tilde{\lambda}'_{-i})$. Thus, $\varphi^{\mathbf{C}}(\tilde{\lambda}'') = \varphi^{\mathbf{C}}(\tilde{\lambda}')$. Let $\tilde{\lambda} \in \mathcal{D}^{\Pi}(\mathbf{C})$ be such that $D(\tilde{\lambda}_i) = \{\omega_j\}$, $D(\tilde{\lambda}_j) = \{\omega_i\}$, and and for each patient $k \notin \{i, j\}$, $\tilde{\lambda}_k = \tilde{\lambda}''_k$. By *individual rationality* and *strategy-proofness*, $\varphi_i^{\mathbf{C}}(\tilde{\lambda}_i, \tilde{\lambda}''_{-i}) = \{\omega_i\}$. By *individual rationality*, $\varphi_j^{\mathbf{C}}(\tilde{\lambda}_i, \tilde{\lambda}''_{-i}) \in \{\omega_j, \varphi_j^{\mathbf{C}}(\tilde{\lambda}'')\}$. By *2-efficiency*, $\varphi_j^{\mathbf{C}}(\tilde{\lambda}_i, \tilde{\lambda}''_{-i}) = \varphi_j^{\mathbf{C}}(\tilde{\lambda}'')$. Finally, by *individual rationality* and *strategy-proofness*, $\varphi_j^{\mathbf{C}}(\tilde{\lambda}_{\{i,j\}}, \tilde{\lambda}''_{-\{i,j\}}) = \{\omega_j\}$ and $\varphi_i^{\mathbf{C}}(\tilde{\lambda}_{\{i,j\}}, \tilde{\lambda}'_{-\{i,j\}}) = \{\omega_i\}$, which violates *2-efficiency*. \square

Example 1'. Let $N = \{1, 2, 3, 4\}$, $\Pi(1) = \{\omega_1, \omega_2\}$, $\Pi(2) = \{\omega_3, \omega_4\}$, and \mathbf{C} such that:

$$\begin{aligned} \mathbf{C}_1 &= \{\omega_2, \omega_3, \omega_4\}, & \mathbf{C}_3 &= \{\omega_1\}, \\ \mathbf{C}_2 &= \{\omega_1, \omega_3, \omega_4\}, & \mathbf{C}_4 &= \{\omega_2\}. \end{aligned}$$

Consider the preference profile $\tilde{\lambda} \in \mathcal{D}^{\Pi}$ such that

$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	$\tilde{\lambda}_3$	$\tilde{\lambda}_4$
ω_2	ω_1	ω_1	ω_2
$\omega_3 \sim_1 \omega_4$	$\omega_3 \sim_1 \omega_4$	ω_3	ω_4
ω_1	ω_2	\dots	\dots

Note that $[(1, \omega_3), (2, \omega_4), (3, \omega_1), (4, \omega_2)] \in \mathcal{I}(\tilde{\lambda})$. Replicating the arguments in the proofs of Lemmata 1 and 1', we obtain that for every rule $\varphi^{\mathbf{C}} : \mathcal{D}^{\Pi}(\mathbf{C}) \rightarrow \mathcal{A}^2$ that satisfies individual rationality, 2-efficiency, strategy-proofness, and non-bossiness,

$$\varphi^{\mathbf{C}}(\tilde{\lambda}) = [(1, \omega_2), (2, \omega_1), (3, \omega_3), (4, \omega_4)].$$

Corollary 1. *There exists \mathbf{C} such that if a rule $\varphi^{\mathbf{C}}$ satisfies individual rationality, 2-efficiency, strategy-proofness, and non-bossiness, then for some preference profile $\succsim \in \mathcal{D}^{\Pi}(\mathbf{C})$, $\varphi^{\mathbf{C}}(\succsim)$ does not maximize the number of mutually compatible kidney exchanges.*